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Quests for a General Method to Solve the Equations of High Degree in the History of Islamic Mathematics: The Case of Irshād al-ṭullāb ilā ʿilm al-ḥisāb

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Abstract: This article introduces a book titled *Irshād al-ṭullāb ilā ʿilm al-ḥisāb*, written by an anonymous author who most probably lived in Ottoman lands, presented the book to Ottoman Sultan Bayezid II. In the epilogue of the book, the author presents an equation solving method, unknown up to its period according to our research, which he used to solve third, fourth, fifth and sixth degree equations whose degrees of variables are successively ordered. Following a short introduction on the subject, I first summarize the history of solving methods for third and higher degree equations, and then I examine the mathematical analysis of the book's epilogue. After a short conclusion, I present the Arabic text of the manuscript and its Turkish translation.

Keywords: Equations of High Degree, Algebra in the Ottomans, Mathematics in the Ottomans, Equations Solving Methods, Irshād al-ṭullāb.

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Introduction

The history of Mathematics in the Muslim world usually begins in third/ninth century. Both Arabic translations of the books on mathematics inherited from previous civilizations and the establishment of the new branches of Mathematics based on existing knowledge began to appear in this century. Perhaps, the most important of these new branches was algebra on which al-Khwārizmī for the first time wrote an independent and systematic book. Algebra is derived from the word *al-jabr* mentioned in that book's title. The book explains how to find known and unknown values through certain methods. The author demonstrates six kinds of equations (*masāil-i sitta*), three of which are linear (first degree) and the rest are quadratic (second degree). He also explains conversion methods of quadratic equations that fall out of these six categories which could be reduced to one of the six. In addition to al-Khwārizmī's findings, his successors sought to find analytical solutions for cubic or higher degree determinate and indeterminate equations that could be converted into one of six equation categories. We observe that mathematicians in the Muslim world from fourth/tenth centuries onward continued to work significantly on arithmetical and geometrical solution methods for higher degree equations that cannot be converted into basic equation forms. Related to this, they even advanced in finding rational and irrational roots of integers and rational numbers, extracting square roots of polynomials, demonstrating proofs of solutions for third degree equations through conic section and first algebraic abstraction of object problems.[1] One of these mathematicians was the author of *Irshād al-ṭullab ilā ʿilm al-ḥisāb* (Guide to students of the science of calculation). The book, presented to Sultan Bayezid II (1481-1512), is representative of the ninth/fifteenth century Ottoman mathematics and was a natural continuation of the tradition of mathematics in Islamic civilization.

As a general mathematics textbook, it includes calculations of integers and rational numbers, algebra and area measurement.[2] The distinguishing aspect of *Irshād al-ṭullāb* from other contemporary mathematics books and the reason why I chose it as a subject in this article, pertains to the second part of the third chapter. In these sections on algebra, the author provides some conclusions on the six basic equation forms. According to this, the author states that the six basic equation forms are applicable only to the problems convertible to one of them and problems that are inconvertible require another method of solution for which as he promises to explain in the epilogue.[3]

1 The principal works on the subject are Al-Karajī's *al-Fakhrī fi al-jabr wa al-muqābala* and *al-Badīʿ fī aʿmāl al-ḥisāb*, ʿUmar Khayyam's *al-Maqāla fi al-jabr wa al-muqābala*, Sharaf al-Dīn al-Ṭūsī's *al-Muʿādalāt*, Nizam al-Dīn Nīshābūrī's *al-Shamsiyya fi al-ḥisāb*, Ibn al-Hāim's *al-Mumtiʿ fī sharḥ al-Muqni*, and Ali Qushjī's *al-Muḥammadiyya fi al-hisāb*.

2 For a detailed examination of the book see. Fazlıoğlu, "İrşādu't-Tullāb İlā İlmi'l-Hisāb", 315-340.

3 *İrshād al-ṭullāb*, 64b-65a.

I. The Origins and Development of High Degree Equations

While the origin of linear and quadratic equations goes back to 2000 BC, arithmetical and geometrical solution methods for third degree equations appeared from fourth/tenth century. Since geometrical solutions and proofs were not applicable to fourth and higher degree equations, researches for arithmetical solution methods began almost during the same period. Although Diophantus from the third century gives some examples of fourth degree equations in his *Aritmetika*,[4] he mostly limited his work to indeterminate equations and used numerical analysis instead of algebra. Thus, we can say in light of the following research that real attempts toward further understanding this subject first appeared in Islamic civilization.

Thābit b. Qurra (died 288/901), a successor of Al-Khwārizmī, managed to solve a cubic equation by finding intersection points of a circle and a hyperbole through a method similar to ʿUmar Khayyām's development of positive roots of cubic equations a century and a half later.[5] Māhānī (died c. 266/880), a contemporary of Thābit b. Qurra converted a solid matter problem of "how to divide a plane and a sphere into two equal parts with known volumetric proportions" into the third degree equation of , known as "Māhānī equation". Khāzin (died c. 360/971) solved this equation by using cone sections, thereby paved the way for geometric solutions for third degree equations.[6] ʿUmar Khayyām (died 525/1131), who established the basics of analytical geometry by saying that algebra proves geometrical realities, added 13 categories of third degree equation to the six basic categories and presented a new classification composed of 19 equation forms.[7] Sharaf al-Dīn al-Ṭūsī from sixth/twelfth century advanced his predecessor Khayyām's works and increased the equation categories to twenty-five. He researched on the equation roots through a method used for finding dividing polynomials and extracting their roots, which is known as Ruffini-Horner method whose origin goes back to the Karajī school.[8] The work of these analytical geometricians led to the spread of third degree equations and encouraged mathematicians to pursue this trajectory. However, they rejected a fourth dimension as well as fourth and higher degree equations because they used to extract continuous quantities by the way of abstraction from external world, which may have had a negative influence on the development of algebra in this issue.

4 Diophantus, *Ṣināʿat al-jabr*.

5 Rosenfeld and Grigorian, "Thābit Ibn Qurra", 291; Rāshid, *Mawsūʿāt*, II, 468.

6 Dold-Samplonius, "al-Māhānī", 21; Rāshid, *Mawsūʿāt*, II, 468-469.

7 Khayyām, *al-Maqāla fi al-jabr wa al-muqābala*, 5-13.

8 Rāshid, *el-Jabr wa al-handasa*, 91-160, 176-177, 450-680.

Following our discussion on the development of higher degree equations in the context of analytical geometry, we can now examine arithmetic algebra, another orientation in the history of algebra. Firstly, arithmetic algebraists' efforts, unlike geometric algebraists who limited themselves to third degree equations, formed equations at higher degrees as much as possible and discovered various methods in order to find exact or approximate results.

We should acknowledge the contributions of al-Karajī (died after 410/1019), who is known as the renewer (*mujaddid)* of algebra because he laid down the foundations of arithmetic algebra, and the Karajī school named after him advanced arithmetic algebra by developing algebra through arithmetic. Sinān b. Fatḥ from third/ninth century, with his work titled *Risāla fī al-ka'b wa al-māl wa al-a'dād al-mutanāsiba*, initiated the first studies that would pave the way for third and higher degree equations.[9] Al-Karajī reexamined the second-degree equations and sought to solve higher-degree equations by converting them into second-degree equations. His successors advanced in direct analysis of third and fourth-degree equations without converting them.[10] Samaw'al al-Maghribī (died 575/1180), one of the Karajī school's most important members, made remarkable progress by dividing polynomials and finding polynomial roots that are crucial in solving high-degree equations, such as: and .[11] He converted a polynomial with twelve terms into a polynomial with four terms and also a polynomial with eight terms into a polynomial with three terms. He managed to do these long and considerably hard operations very practically and simply with the help of tables.[12] One of the sources containing plenty of examples of high-degree equations was *al-Fawā'id al-Bahā'iyya fī al-qavā'id al-ḥisābiyya* of Ibn al-Khawwam (died 724/1324), a person who was known as the founder of Ottoman Mathematics. He also cited in the epilogue of *Irshād al-ṭullāb*.[13] Based on our research the work on pure algebra, *al-Mumti' fī sharḥ al-Muqni'*, of Ibn al-Hāim (died 815/1412) was influential in transmitting the Maghrib-Egyptian tradition of Mathematics to the Ottoman Empire. In that work, he presents the most comprehensive knowledge on high-degree equations and solution methods of the time. However, although some these equations included constant term, although some of them did not, as different from those high-degree equations in the forms of that the author of *Irshād* sought to solve. In addition, while Ibn al-Hāim used some

9 For Sinān b. Fatḥ and his work see Rāshid, *Ta'rīkh al-riyāḍiyyāt*, 24-25, 31; Rāshid, *Mawsū'āt*, II, 469; İhsan Fazlıoğlu, "Harranlı Matematikçilerin Matematiğin Oluşumundaki Katkıları".

10 Rāshid, *Mavsū'āt*, II, 473; Rāshid, *Ta'rīkh al-riyāḍiyyāt*, 35; Saidan, *Ta'rīkhu 'ilm al-jabr*, 83; Rāshid, "al-Karaji", 242.

11 Anbouba, "al-Samaw'al", 92-93; Rāshid, *Mawsū'āt*, II, 472; Fazlıoğlu, "Semev'el-Mağribī", 490-495.

12 Maghribī, *al-Bāḥir fī al-jabr*, 44-50.

13 For the author and his work see Fazlıoğlu, *İbn el-Havvām ve Eseri*; Fazlıoğlu, "İbn el-Havvām, Eserleri ve el-Fevāid el-Behāiyye".

methods which are also included in modern mathematics, such as conversion, substitution, assumption etc., *Irshād*'s author employed a completely different method. The author's emphasis on his unique, unprecedented, yet confusing method was probably related to this difference.[14]

When we look at the examples of high-degree equations from aforementioned Mathematicians, we can see that al-Karajī's equations reaches the seventh-degree that are solved by converting them into one of six basic forms through simplification and substitution. Let us give some examples:[15]

Example 1:

$$x^4 + 5x^2 = 126 \quad \Rightarrow \quad \textit{if we assume } x^2 = t, \qquad \textit{the equation becomes}$$

$$t^2 + 5t = 126 \ \textit{and} \ x^2 + \text{bx} = \text{c} \quad \Rightarrow \text{ since } x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}$$

$$\textit{then } t = \sqrt{\left(\frac{5}{2}\right)^2 + 126} - \frac{5}{2} = \sqrt{6 + \frac{1}{4} + 126} - \frac{5}{2} = \sqrt{132 + \frac{1}{4}} - \frac{5}{2} = 11 + \frac{1}{2} - \frac{5}{2} = 9$$

$$\Rightarrow \text{as we assume } x^2 = t \ \textit{then} \ \boldsymbol{x^2 = 9} \quad \Rightarrow \quad \boldsymbol{x^4 = 81}$$

Example 2:

$$x^5 + x^3 = x^7 \quad \Rightarrow \quad \frac{x^5}{x^3} + \frac{x^3}{x^3} = \frac{x^7}{x^3} \quad \Rightarrow \text{ it becomes } x^2 + 1 = x^4 \ \textit{if we assume}$$

$$x^2 = t, \textit{the equation becomes } t + 1 = t^2. \quad bx + c = x^2 \quad \Rightarrow$$

$$\text{as } x = \sqrt{\left(\frac{b}{2}\right)^2 + c} + \frac{b}{2} \ \ \textit{then } t = \sqrt{\left(\frac{1}{2}\right)^2 + 1} + \frac{1}{2} = \sqrt{\frac{5}{4}} + \frac{1}{2} \Rightarrow x^2 = \sqrt{\frac{5}{4}} + \frac{1}{2}$$

For the equations that Ibn al-Havvām could not solve or did not know the possibility of having a solution, we can give such an example:[16]

Example 1: We want to find a cube integer. The difference between this cube integer and its square is also a square integer.

$(a^3)^2 - a^3 = b^2 \Rightarrow a^6 - a^3 = b^2$ veya $a^3 - (a^3)^2 = c^2 \Rightarrow a^3 - a^6 = c^2$

14 *Irshād al-ṭullāb*, 4a-4b.

15 Karajī, *Kitāb al-Fakhrī li al-Karajī*, cf. Saidan, *Ta'rīkhu 'ilm al-jabr*, I, 164-165.

16 Ibn al-Khawwām, *al-Fawā'id al-Bahā'iyya*, 99a.

Example 2: We want to find a square integer. If we multiply with itself and add 10 times of its square root and 10, we will find square integer.

$(a^2 . a^2) + 10a + 10 = b^2 \Rightarrow a^4 + 10a + 10 = b^2$

We can give three examples from Ibn al-Hāim who was also interested in high-degree equations. We can also state that he had a considerably important position in the Muslim world thanks to ubiquity of his works:

Example 1:[17]

$$20x^3 = 5x^4 + 2x^5 + \frac{x^5}{2} \quad \Rightarrow \quad 5x + 2x^2 + \frac{x^2}{2} = 20 \quad \Rightarrow \quad x = 2, x^2 = 4, x^3 = 8,$$

$$x^4 = 16, \ x^5 = 32, \ 2x^5 + \frac{x^5}{2} = 80, \ 5x^4 = 80, \ 20x^3 = 160$$

Example 2:[18]

$$\boldsymbol{x^4 + 2x^3 = x + 30} \Rightarrow (x^2 + x)^2 = x^4 + 2x^3 + x^2 \ \Rightarrow \ x^4 + 2x^3 + x^2 = x^2 + x + 30$$

$$\Rightarrow \ (x^2 + x)^2 = x^2 + x + 30 \ \Rightarrow \ \textit{we assume } \boldsymbol{x^2 + x = y} \ \textit{ then} \ \Rightarrow \ y^2 = y + 30 \ \Rightarrow$$

$$y = 6 \Rightarrow x^2 + x = 6 \quad \Rightarrow \quad \boldsymbol{x = 2}, \ x^2 = 4, \ x^3 = 8, \ x^4 = 16 \ \Rightarrow$$

$$x^4 + 2x^3 = 16 + 16 = 32$$

Example 3:[19]

$$x^7 = 28x + 4x^4 + \frac{x^4}{2} \Rightarrow \text{the exponents are successively ordered by 3: } 7, 4, 1$$

$$\Rightarrow \text{if we reduce the exponents to } 2, 1, 0 \ \textit{the equation becomes} \ \Rightarrow x^2 = 28 + 4x + \frac{x}{2}$$

$$\Rightarrow \text{then } x == 8 \ \textit{and because they are successive by } 3, \textit{originally it becomes } \boldsymbol{x^3}$$

$$= \boldsymbol{8} \ \boldsymbol{ve} \ \boldsymbol{x = 2}.$$

Having examined some of the examples for high degree equations from the history of Mathematics in the Muslim world, it is possible to contextualize the uniqueness of the *Irshād* and its author by way of modern mathematical symbols.

II. Mathematical Analysis

Before moving into mathematical analysis, I will first give a short summary of the chapter. The author starts the chapter directly with the rules related to his meth-

17 Ibn al-Hāim, *al-Mumti*ʿ, 77b.

18 Ibn al-Hāim, *al-Mumti*ʿ, 78b

19 Ibn al-Hāim, *al-Mumti*ʿ, 77b-78a.

od because he already outlined general rules and principles in the third chapter on algebra. He first states that equations are made of known terms, i.e. numbers, and unknown terms and also looks at the exponents and roots of these terms. Then, he requires the condition that the exponents should be successively ordered by 1, 2 or 3 etc. in any side of the equation in order to use his method. He calls right and left sides of the equation with certain names according to their highest exponents. He presents different versions of solution formula in respect to whether they have fractional numbers or whether any side has only one term. Lastly, he explains the application of his method to several problems one of which is from Ibn al-Khawwām, a scientist from seventh/thirteenth-eight/fourteenth centuries, another of which is an inheritance problem.

1. Conditions of Values (a, a^2, a^3 ...)

When a number is multiplied with its first root (*ḍıl'*), the result is the number whose exponent is greater by one degree of the exponent of the former. The operation is the opposite in division (i.e. when its first root divides a number, the result is the number whose exponent is less by one degree of the exponent of the former). That the author emphasized successively increase or decrease in the base of a number when it is multiplied or divided by their first roots might have had to do with making the reader familiar with kinds of equations whose solution he will provide.[20]

$$\forall a \in \mathbb{R}\backslash\{0\} \text{ and } m,n \in Z^+ \Rightarrow a^n.a^n = a^{2n} \text{ and } a^m.a^n = a^{m+n}, \text{thus}:$$

$$a.a = a^2$$

$$a^2.a = a^3$$

$$a^3.a = a^4$$

$$a^4.a = a^5$$

$$\infty$$

$$\forall a \in \mathbb{R}\backslash\{0\} \text{ and } m,n \in Z^+ \text{ and } n > m \Rightarrow \frac{a^n}{a^n} = a^{n-n} = a^0 \text{ ve } \frac{a^n}{a^m} = a^{n-m}$$

$$\infty$$

$$\frac{a^5}{a} = a^{5-1} = a^4$$

$$\frac{a^4}{a} = a^{4-1} = a^3$$

$$\frac{a^3}{a} = a^{3-1} = a^2$$

$$\frac{a^2}{a} = a^{2-1} = a^1$$

$$\frac{a}{a} = a^{1-1} = a^0$$

20 *Irshād al-ṭullāb*, 111a-111b.

2. Definitions

The author states that the solution he offers can be used in certain kinds of equations where exponents of terms in both sides of the equation may successively be ordered, thereby he suggests that one should check whether the equation satisfies this condition before solving it. If it satisfies this, one should look at the exponents of the terms in the right and left sides, then the side having a term with a higher exponent becomes *maqīs alayh* (comparing side) and the other side becomes *maqīs* (compared side). Thus, one can avoid a possible confusion in the next operation.[21]

$\forall$ *a, b, c, d, e*

$\in \mathrm{R}\backslash\{0\}$ ve *m,n,p,q are positive successive integers*

$\Rightarrow$ *in the equation of* $ax^m + bx^n = cx^p + dx^q + e(x^0)$; $m,n>p,q$

$\Rightarrow$ left side of the equation $\Rightarrow$ muqīs alayh,and the right side $\Rightarrow$ muqīs

3. The method for high-degree equations

One can start to solve equations after defining the sides. But one should look first whether terms have integers or fractional numbers, because the formula varies in each condition.

3.1. The method for the equations that do not include fractional numbers in the terms

If there are only integers in an equation and if there are more than one term in both sides of the equation, the solution set for the problem can be found according the formula below:[22]

$\forall\ b, c, d\ \in \mathbb{R}\backslash\{0\}$ *ve m, n, p are positive successive integers*

$\Rightarrow$ *in the equation of* $x^m + bx^n = cx^p + d(x^0)$,

$m > n > p$ *ve* $m - 0 = m \Rightarrow\ x^m \geq cm^p + d\ \ \Rightarrow$ *then it becomes* $x \cong \sqrt[m]{cm^p + d}$.

Example:[23]

$x^3 + 4x^2 = 10x + 31 \quad \Rightarrow \quad x^3 + 4x^2 = 10x + 31(x^0)$

$\Rightarrow$ *the difference between the highest exponent and the lowest exponent*

$\Rightarrow \quad 3 - 0 = 3 \Rightarrow x^3 \geq 10.3 + 31 \Rightarrow x^3 \geq 61 \Rightarrow \mathrm{x} \geq \sqrt[3]{61} \Rightarrow x \cong 3$

21 *Irshād al-ṭullāb*, 111b.

22 *Irshād al-ṭullāb*, 111b-112a.

23 *Irshād al-ṭullāb*, 113a.

As it is seen in the example, the method does not promise to find a precise and exact result. By his method, one can obtain an exact result as well as an approximate result or an interval of solution set. This formula and the following ones offer a faster and simpler solution through limiting the estimated numbers in order to find the equalizing number when it substitutes the unknown number in the equation.

If we solve the same equation through the method of finding integer roots in an equation in modern Mathematics, then:

$a_1, a_2, \dots, a_{n-1}, a_n \in Z \Rightarrow x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$ *is a n-th degree polynomial equation.*

If a p integer is a root of this equation,then p is a divisor of the constant term a_n.

Example:

if we convert the equation of $x^3+4x^2=10x+31$

into a polynomial equation form $\Rightarrow x^3+4x^2-10x-31=0$

$\Rightarrow$ *the divisors of 31 are* {1,-1,31,-31}dir.

We can see that none of these divisors satisfies the equation when they are applied in the equation, then this equation does not have integer roots. We need to apply the trigonometric method of François Viete (1540-1603) to find other roots. Even if this method gives exact results, it has been known as a long and impractical method.

3.1.1. If the Maqīs has only one term:

If the terms of the equation are integers and one side has only one term, the formula varies a little. If the *maqīs*, the side with lower exponential term, has only one term, the formula is the following: [24]

$$\forall\ b,c,d,e \in \mathbb{R}\setminus\{0\}\ \text{and}\ m,n,p,q\ \text{are positive successive integers}$$
$$\Rightarrow \text{in the equation of } x^m + bx^n + cx^p + dx^q = e(x^0)\ \text{it becomes}\ m > n > p > q$$
$$\Rightarrow m - 0 = m\ \text{thus}\ x^m \geq e \Rightarrow x \cong \sqrt[m]{e}\ \text{or}\ x \cong \sqrt[m]{e} - 1$$

Example:[25]

$$x^4 + 2x^3 + 6x^2 + 5x = 66(x^0)\ \text{thus}\ 4 - 0 = 4 \Rightarrow x^4 \geq 66 \Rightarrow x \cong \sqrt[4]{66}\ \text{or}\ x \cong \sqrt[4]{66} - 1 \Rightarrow x \cong 3\ \text{or}\ x \cong 2 \Rightarrow \text{since}\ x = 2\ \text{satisfies the equation,}$$
$$\text{the solution set} = \{2\}$$

24 *Irshād al-ṭullāb*, 112a.
25 *Irshād al-ṭullāb*, 113a-113b.

According to the formula, the found numbers in the solution set should be applied in the equation. Then:

$$x^4 + 2x^3 + 6x^2 + 5x = 66(x^0) \; thus \quad x = 2 \;\Rightarrow\; 2^4 + 2.2^3 + 6.2^2 + 5.2 = 66$$
$$\Rightarrow \quad 16 + 16 + 24 + 10 = 66 \;\Rightarrow\; 66 = 66$$

The author states, following his discussion of this solution, that Ibn al-Khawwām (from 7th/13th-8th-14th centuries) also presented the equation in a different way. But we do not see any similar example in the chapters on Algebra or unsolvable problems/equations in Ibn al-Havvām's extant work on unsolvable problems, titled *al-Fawā'id al-Bahā'iyya fi al-qawā'id al-ḥisābiyya*. It is also possible however that the anonymous author may have consulted to another work of Ibn al-Khawwām that we do not have today. The author's citation shows the importance of both Ibn al-Khawwām and the traditions of unsolvable problems in Ottoman Mathematics, even if this has not been clearly verified. Here is the problem referred by Ibn al-Khawwām and the conversion of this problem into an equation in *al-Irshād*:[26]

$$\left(\sqrt{x^2} + \sqrt{\sqrt{x^2}} + 3\right).\left(2\sqrt{x^2} + 2\sqrt{\sqrt{x^2}} + 4\right) = 144 \quad \Rightarrow \quad if\ we\ assume\ x^2$$
$$= y^4\ then\ it\ becomes \;\Rightarrow \left(\sqrt{y^4} + \sqrt{\sqrt{y^4}} + 3\right).\left(2\sqrt{y^4} + 2\sqrt{\sqrt{y^4}} + 4\right) = 144\,.$$
$$\Rightarrow\; (y^2 + y + 3).(2y^2 + 2y + 4) = 144 \;\Rightarrow$$
$$2y^4 + 4y^3 + 12y^2 + 10y + 12 = 144 \;\Rightarrow\; y^4 + 2y^3 + 6y^2 + 5y = 66$$

Having presented the equation, *al-Irshād's* author quotes one of Ibn al-Havvām's sentences guiding in problem solving:

ثم قال: وإستخراج الجواب في مثل هذه المواضع إنّما يكون بالفكر والطلب والتتبع. فإنه ما لنا طريق يستخرج به الشيء في مثل هذه الصور.

Then [Ibn al-Khawwām] said: "Finding an answer in these questions can only be possible through thought, demand and research. We do not have however any method finding an unknown as seen in these problems."[27]

If we need to explain this statement, we should assert that three actions would lead us to solution when encountered a problem. The first action is to determine the category of the problem among three kinds of 'imperative,' 'possible' or 'impossible;'

26 *Irshād al-ṭullāb*, 113b-114a. See also İhsan Fazlıoğlu, "İbn El-Havvām, Eserleri", 109-110.

27 *Irshād al-ṭullāb*, 114a.

and, unless it is 'impossible,' to think on what kind of problem it is and on what kind of a solution method we need to apply. Secondly, we need to understand clearly what has been asked and targeted in the problem and lead the solution process accordingly. Thirdly, we should follow the sequence and order in the operations from the beginning to the end.

To show the solution of the equation through modern Mathematics, let us turn the equation into a polynomial form, i.e. one side of the equation having all unknown and constant terms and the other side having only zero.

$y^4+2y^3+6y^2+5y=66 \Rightarrow y^4+2y^3+6y^2+5y-66=0$

Since the divisors (2 and -3) of the equation's constant term (-66) satisfy the equation, then $\{y_1=2$ and $y_2=-3\}$. Let us use these roots to find other roots:

$y^4+2y^3+6y^2+5y-66=(x-2).(y^3+4y^2+14y+33) \Rightarrow$

So $y-2=0$ or $y^3+4y^2+14y+33=0$

We already know the first equation is the first root. Only the divisor (-3) of the constant term of the second equation satisfies the equation. Then:

$y^3+4y^2+14y+33= (y+3).(y^2+y+11) \Rightarrow$

$y+3=0$ or $x^2+x+11=0$.

We know the first equation is second root. Then in the second equation:

from $\Delta=b^2-4ac$' we find $\Delta<0$ çıkar.

Thus, the equation has no reel number root but only complex number root. Consequently:

$y^4+2y^3+6y^2+5y-66=(y-2).(y+3).(y^2+y+11) \Rightarrow$ Solution set= $\{2,-3\}$

3.1.2. If maqīs alayh has only one term:

When the *maqīs*, the side of the equation where a variable with higher exponent exists, has only one term, we apply the following formula.[28]

$$\forall\, b,c,d,e \in \mathbb{R}\backslash\{0\} \text{ and } m,n,p,q \text{ are positive successive integers } \Rightarrow \text{since,}$$
$$\text{in the equation of } x^m = bx^n + cx^p + dx^q + e(x^0),$$
$$\text{it becomes } m > n > p > q \Rightarrow \quad m - 0 = m, \qquad \text{then } x^m \le e \quad \Rightarrow \quad x \le \sqrt[m]{e}$$

28 *Irshād al-ṭullāb*, 112a-112b.

Example:[29]

In the equation of $x^3 = 3x^2 + 16(x^0)$ *it is* $3 - 0 = 3 \Rightarrow x^3 \leq 16 \Rightarrow$

$x \leq \sqrt[3]{16} \Rightarrow x \cong 2$

In this example the author makes a mistake. The solution set does not satisfy the equation. He presents either the problem/equation inaccurately and incompletely or he applies an incorrect solution. The first possibility seems to be true, because the author does not follow his own rule, specified at the beginning, that the exponents of the variables should be ordered successively.

If we solve the problem through methods we know today, then:

$x^3=3x^2+16(x^0) \Rightarrow x^3-3x^2-16=0$

Since only the divisor (4) of the constant term (16) satisfies the equation and when the equation is divided by (x-4) it becomes:

$x^3-3x^2-16=(x-4).(x^2+x)+4x-16$

$x^3=3x^2+16(x^0)$ the real solution set is $=\{4\}$.

3.2. A solution method when fractional numbers exist in the equation's terms

Since the author is ambiguous in the conclusion of the book, and his method does not always lead to correct results, it is somewhat difficult to present the method through a mathematical language where everything needs to be precise and exact. One of the difficulties is that the author does not follow the rules in his presentation nor in his solution. He presents examples without checking whether the method is sufficient to solve the equation. These difficulties are especially acute when solving equations with fractional numbers.[30] For this, the author explains it through a complicated inheritance problem and science of *farāiḍ* (inheritance law), which is one of the most important areas of application for Algebra. Because of this, I will not give the solution formula here, but look at the inheritance problem and its solution. We will first form an algebraic equation to solve the problem, then solve this equation with fractional numbers through the author's method.

29 *Irshād al-ṭullāb*, 114a.

30 *Irshād al-ṭullāb*, 114a-115a.

An example for an inheritance calculation[31]

The deceased man left behind a wife, three sons and a daughter and he also bestowed to his maternal and paternal uncle some share for each one. The share of paternal uncle is to be twice as much the share of maternal uncle, then the sum of these two shares will be 1/10 of the 3/5 of the whole inherited wealth. The wife's share equals to the value we would have when we add 2 and 5/8 to the cube of the 2/3 of the subtraction of the multiplication of uncles' shares from the sum of their squares.

Fariza (natural share holders) 1 wife, 3 sons, 1 daughter

Bestowal 1 share for paternal uncle and 1 share for maternal uncle

Maternal uncle's share = x

Paternal uncle's share = 2x

Inherited wealth = y,

Since the sum of uncles' shares equals to 1/10 of 1/10 of 3/5 of the whole inherited wealth.

$$2x + x = y.\frac{3}{5}.\frac{1}{10}.\frac{1}{10} \quad \Rightarrow \quad 3x = \frac{3y}{500} \quad \Rightarrow \quad then\ y = 500x$$

The wife's share:

$$\left[\frac{2\left((x^2+4x^2) - 2x^2\right)}{3}\right]^3 + 2 + \frac{5}{8}$$

$$= \mathbf{8x^6} + \frac{\mathbf{21}}{\mathbf{8}} \quad so, the\ equation\ for\ calculating\ the\ inherited\ shares\ becomes$$

$$64x^6 + 21 = 500x \ \ and\ when\ we\ divide\ both\ sides\ by\ 64\ the\ equation\ becomes$$

$$x^6 + \frac{21}{64} = 7x + \frac{3x}{4} + \frac{x}{16}.$$

According the author's formula for solving a high-degree equation with fractional numbers, we should take the highest root, defined by highest exponent, of highest denominator : $\{\sqrt[6]{64} = 2\}$ and convert the fractional numbers into integers by 2,the root of proportion (ḍıl' al-nisba):

$$then\ it\ becomes\ x^6 + 21 = 250x\ .The\ difference\ between\ the\ exponents\ \ 6 - 1 = 5$$

$$\Rightarrow\ x^5 \geq 250 \ \Rightarrow\ x \geq \sqrt[5]{250} \ \Rightarrow\ \boldsymbol{x \geq 3} \ \Rightarrow\ 3^6 + 21 = 250.3 \ \Rightarrow\ 729 + 21 = 750$$

31 *Irshād al-ṭullāb*, 115a-116a.

Since we converted the original equation with fractional numbers into an equation with integers by 2 as the root of proportion, while it is

{x=3} in this new equation,it must be $\left\{x = \frac{3}{2} = 1 + \frac{1}{2}\right\}$ in original equation.

Conclusion and Evaluation

I have arrived at the following conclusions based on my mathematical analysis of the solution for the equations inconvertible into any of the six basic equation forms presented in the epilogue of a Mathematics book, titled *Irshād al-ṭullāb ilā ʻilm al-ḥisāb*:

1. Although the method offers various formulas in respect to the conditions of the equation's sides and to the existence of integers or fractional numbers of constant terms, it does not always give accurate and precise solution.
2. The method aims at limiting the quantity of numbers in a solution set rather than finding the precise solution and thereby at simplifying the solution process.
3. Some of the mistakes in the given examples suggest that the author might have presented them without checking the equations and solutions or copied them from another source without controlling their verifications.
4. The author's ambiguous statements especially in the epilogue make understanding of the method unsupported by precise and substantial mathematical proofs much more complicated and lead us to think that even he is not so sure about the method.
5. The fact that he gives one of the examples from calculating an inheritance sharing shows that the method goes beyond a theoretical concern and aims at simplifying problems in everyday life. This aim is related to the purpose of the Ottoman tradition of science in widening the area of usages for theoretical sciences, especially Mathematical sciences, by employing them together with the applied sciences and thereby extracting maximum benefit from all.
6. Despite its deficiencies and mistakes, the method should be seen as an important contribution to the solutions for high-degree equations since the early period in ninth/fifteenth century. As for developing a general method giving precise solutions to third and fourth-degree equations, we should wait another two centuries. Even if it is still available today, modern mathematicians find this method impractical. As for fifth and higher-degree equations, it has been now proved that no method giving precise solutions is available.

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الخاتمة: فيما سبق الوعد به[1]

فنقول أوّلاً من المعلوم في تركيب الأنواع: إنّ مقدار كل مفرد إذا ضرب[2] في ضلعه، خرج مقدار النوع المفرد الذي يليه بعده، وقسمته أيضا على الضلع، يخرج النوع الذي يليه قبله. فإذا في كل نوع مفرد من مفردات النوع الذي يليه قبله بقدر ما في الضلع الواحد من الآحاد، فإذا تحقّق ذلك، فانظر إلى العديلين، فأيّهما اشتمل على أعلى المراتب، فسمّه المقيس عليه والآخر المقيس. ثمّ خذ الفضل بين أس الأ أعلى من أحدهما وبين أس الأدنى من الآخر، واطلب أقل عدد من جنس الفضل، واضربه في عدد أعلى مراتب المقيس، وضم الحاصل إلى ما تجاور[3] من المراتب إن كان، وضرب الحاصل أيضا في ذلك العدد، وجمع الحاصل إلى ما تجاوره[4] من المراتب إن كان، وهكذا إلى[5] أن يصل إلى الأدنى. فإن تساوي المجتمع حاصل ذلك العدد أو زاد عليه، فالعدد الذي يلي ذلك العدد قبله هو نهاية أعداد التخمين. وكذا، إن تساويا وكان الأعلى غير مفرد وإلاّ فهو ذلك العدد بعينه.

فإن كان المقيس مفردا، فخذ فضل ما بين أسه وأس الأعلى وأطلب أقل من جنس الفضل يساوي عدد المفرد أو يزيد عليه، فما كان فالعدد الذي يليه بعده هو نهاية أعداد التخمين.

وإن كان المقيس عليه هو المفرد، أخذت الفضل بين أسه وأس الأعلى من المركب، وطلبت أعظم عدد من جنس الفضل يساوي عدد أعلى المركب أو ينقص عنه، فما كان فالذي يليه وما بعده في جهة الرفع هو المطلوب.

فإذا[6] انحصرت الأعداد، فاختبرها واحدا بعد واحد. وطريقه أن تضرب أحدها في عدد أعلى

١ ١١١و في النص.
٢ ١١١ظ في النص.
٣ تجاور: «تحاور» في الناص.
٤ تجاوره: «تحاوره» في الناص.
٥ ١١٢و في النص.
٦ ١١٢ظ في النص.

المراتب، فما حصل فهو نوع أنزل من الأعلى برتبة[٧]. فإن كان مع الأ أعلى جنس آخر من تلك الرتبة[٨]، فاجمع الخارج[٩] إليه وإلا فلا. ثم اضرب المجتمع أو ما صار إليه بالجمع في ذلك العدد أيضا، وافعل بالخارج[١٠] ما تقدّم وهكذا إلى أن يصير المبلغ جنس مساويا لأنزل المراتب الواقعة في أحد العديلين، واحفظ المجتمع. ثمّ اضرب عددا على مراتب العديل الآخر في ذلك العدد، واصنع فيه ما تقدّم الى الآخر، وانظر بين الحاصلين، فإن تساويا، فذلك العدد هو الجذر[١١] المركّب منه تلك الأنواع المفروضة وإلاّ[١٢] فهو أقل. إن زاد[١٣] حاصل المقيس عليه على حاصل المقيس، وإلاّ فهو أكثر. هذا كلّه بعد أن تقسم جميع ما معك أعلى عدد الأعلا. إن تعدّد، فيرجع إلى واحد والباقي على تلك النسبة كما مرّ.

فلو قيل: كعب وأربعة أموال تعدل عشرة أشياء و أحدا وثلاثين درهما، فالفضل بين الأسين ثلاثة. فاطلب أقل عدد، إذا كعب يساوي أو يزيد على حاصل العديل المقيس بعد تحويل جميع ما فيه إلى العدد، تجده ثلاثة وهي توافق عند الامتحان فهي الجذر.

ولو قيل؛ مال مال وكعبان وستة أموال وخمسة أشياء يعدل ستة[١٤] وستين درهما، فالفضل أربعة. فاطلب أقل عدد، إذا صُيِّر مال مال سُوًى حاصله ستة[١٥] وستين أو يزيد عليها، فتجده ثلاثة، فالإثنان هما المطلوب.

وهذه المسئلة بعينها ما وقعت للمولى المعظّم جمال الحق والدين عبد الله بن محمد الخوام البغدادي وصورتها:

أيّ مال إذا ضرب جذره وجذر جذره وثلاثة دراهم في جذريه وجذري جذره وأربعة

٧ برتبة: «برتبه» في النص.

٨ الرتبة: «الرتبه» في النص.

٩ الخارج: «الحارح» في النص.

١٠ بالخارج: «بالخارح» في النص.

١١ الجذر: «الجدر» في النص.

١٢ ١١٣و في النص.

١٣ زاد: «راد» في النص.

١٤ ستة: «سته» في النص.

١٥ ستة: «سته» في النص ؛ ١١٣ظ في النص.

دراهم، كان مائة وأربعة وأربعين. فرضناه مال مال فيكون جذره[١٦] مالا وجذر جذره شيئاً ويكون جذراه مالين وجذرا جذره شيئين. فتضرب جذره وجذر جذره وثلاثة دراهم في جذريه وجذري جذره وأربعة دراهم فيكون الحاصل مالي مال وأربعة كعاب واثني عشر مالا وعشرة[١٧] أشياء واثني عشر درهما وذلك[١٨] يعدل مائة وأربعة وأربعين درهما.

ثم قال: واستخراج الجواب في مثل هذه المواضع إنما يكون بالفكر والطلب والتتبّع. فإنّه ما لنا طريق يستخرج به الشيء في مثل هذه الصور. (انتهى)

ولو قيل؛ كعب يعدل ثلاثة[١٩] أموال وستة عشر درهما، فالفضل ثلاثة. فاطلب أعظم عدد، إذا كعب يساوي ثلاثة أو يزيد عليها، تجده إثنين وهما يوافقان فهما الجذر.

هذا كله إذا لم يكن في المعادلة كسور. فإن كان، فاعرف الفضل بين أسي أعلى المراتب وأنزلها، إن كانا عن العدد في جهة واحدة، وإلا فاجمعهما، يحصل البعد. ثم حل المخرج تلك الكسور إلى أضلاع[٢٠] أول أو غيرها بحيث تنقسم[٢١] على عدد البعد قسمة متساوية في العدة والعدد أو يزيد عليه بفرد من أفراد خارج القسمة أو أقل أو أكثر، وإلاّ فزد في أضلاعه ما يكمل به المطلوب وواحد من تلك الأقسام هو ضلع النسبة، فاعرف مخرجها وضلعها. ثمّ اقسم جميع ما معك على عدد الأ أعلى، ثمّ اطلب عددا كما تقدّم وكمل العمل إلى آخره، وانظر بين الحاصلين، فإن تساويا، فالعدد المأخوذ هو المطلوب وليس في أجناس المسئلة كسور، وإلاّ فالبسط الأنواع التي معك غير الأ أعلى بأن تأخذ البعد بين أس ذلك الجنس الذي تريد بسطه وأس الأعلا فما حصل، فكرِّرْ ضلع النسبة بقدره، وركِّبْ ذلك بالضرب[٢٢]، فما حصل فاضربه في العدد المطلوب بسطه. فإذا بسطت جميع الأنواع، طلبت نهاية أعداد التخمين كما تقدّم وامتحنت الأعداد المحصورة كما تقدم وقسمت ما وافق على ضلع النسبة، فيخرج الجذر، فتأملّ.

١٦ جذره: «حدره» في النص.
١٧ عشرة: «عشره» في النص.
١٨ ١١٤و في النص.
١٩ ثلاثة: «ثلاثه» في النص.
٢٠ أضلاع: «أصلاع» في النص.
٢١ ١١٤ظ في النص.
٢٢ ١١٥و في النص.

فإن قيل؛ ترك ميت[٢٣] زوجة وثلاثة بنين و بنتا وأوصى لكل واحد من عمّه وخاله بوصية[٢٤]، فكانت وصية[٢٥] العم ضعف وصيّة[٢٦] الخال ومجموعهما ثلاثة أخماس عشر عشر التركة[٢٧]. وإذا طرح مسطحهما من مجموع المربعيهما وكعب ثلثا الباقي وزيد على الخارج درهمان وخمسة أثمان، حصل ذلك مثل نصيب الزوجة. فافرض وصية[٢٨] الخال[٢٩] شيأ ووصية العم شيئين واعمل ما قال، تجد أربعة وستين كعب كعب[٣٠] وأحدا وعشرين درهما يعدلان خمسمائة شيء وبعد القسمة كعب كعب وأحدا وعشرين جزء من أربعة وستين من واحد يعدلان سبعة جذور[٣١] وثلاثة أرباع جذر[٣٢] ونصف ثمنه. فالبعد ستة[٣٣] والمخرج أربعة وستون وأضلاعه منقسمة على عدد البعد، فضلع النسبة إثنان وبعد البسط يرجع المعادلة إلى كعب كعب واحد وعشرين درهما يعدلان مائتين وخمسين شيأ، فالبعد بين الأشياء وكعب الكعب خمسة. فاطلب أقل عدد من جنسها يساوي المائتين والخمسين أو يزيد عليها وذلك أربعة فالثلاثة نهاية[٣٤] أعداد التخمين وهي توافق. فاقسمها على ضلع النسبة، يخرج واحد ونصف وهو[٣٥] الجذر[٣٦]. والله أعلم.

وليكن هذا آخر ما أردته وزُبَد هذا الفن فيما اختصرته ولخّصته[٣٧]. والحمد لله على كماله ونواله والصلاة على سيدنا محمد وآله وصحبه وسلم وحسبنا الله ونعم الوكيل. تمّت هذه الرسالة المباركة بعون الله وحسن توفيقه.

٢٣ ميت: في تحت السطر في النص.
٢٤ بوصية: «بوصيه» في النص.
٢٥ وصية: «وصيت» في النص.
٢٦ وصية: «وصيه» في النص.
٢٧ التركة: «التركه» في النص.
٢٨ وصية: «وصيه» في النص.
٢٩ الخال: «الحال» في النص.
٣٠ ١١٥ظ في النص.
٣١ جذور: «جدور» في النص.
٣٢ جذر: «جدر» في النص.
٣٣ ستة: «سته» في النص.
٣٤ نهاية: في هامش.
٣٥ ١١٦و في النص.
٣٦ الجذر: «الجدر» في النص.
٣٧ ولخّصته: «ولحصته» في النص.