NAZARİYAT

ʿAlī al-Qūshjī ʿs Definition of Number in Terms of Its Sources and Influence*

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Abstract: In this study, ʿAlī al-Qūshjī ʿs 'definition of number' will be analyzed in terms of unity, plurality, one, many, quantity, sum, counting, and related concepts. First, to emphasize the importance of the subject, the discussions of the 'definition of number' in contemporary philosophy of mathematics will be briefly reviewed. Then, ʿAlī al-Qūshjī's approach to the subject will be examined through his works *al-Muḥammadiyya fī al-ḥisāb* and *Sharh al-Tajrīd*, and his thoughts will be analyzed. In addition, for a comparison with ʿAlī al-Qūshjī's approach, Shams al-Dīn Iṣfahānī's commentary *Tasdīd al-Qawāʾid fī sharḥ Tajrīd al-ʿaqāʾid* and Sayyid Sharīf's *Ḥāshiya* will be briefly discussed. The background of the attitude ʿAli al-Qūshjī represented will be built on works by members of the Tabriz school of mathematics-astronomy, especially Niẓām al-Dīn al-Nīsābūrī, Ibn Hawwām, Kamāl al-Dīn al-Fārisī, and Jamāl al-Dīn al-Turkistānī, as well as Abū al-Ḥasan al-Bahmanī and ʿAlī b. al-Gharbī. Additionally, the book *Miftāḥ al-ḥussāb* by Jamshīd al-Kāshī of the Samarkand school of mathematics-astronomy, of which ʿAlī al-Qūshjī was a member, will be addressed. Then, the debate triggered by ʿAlī al-Qūshjī in Istanbul will be traced, focusing on mathematicians such as Fanārīzāde ʿAlī Chalabī and the accounting mathematician Kātib ʿAlāʾ al-Dīn Yūsuf. A brief evaluation will be made of the projections of all these discussions in the work of Taqī al-Dīn Rāṣid.

Keywords: Quantity, unity, plurality, number, Jamāl al-Dīn al-Turkistānī, Kamāl al-Dīn al-Fārisī, Jamshīd al-Kāshī, ʿAlī al-Qūshjī

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Introduction

"What is number?" A question framed as "What is...?" is undoubtedly an inquiry into essence and thus ultimately calls for a definition. The concept of number, which is the subject of the question "What is number?", is one of the most intricate concepts that the philosophy of mathematics has studied and subjected to conceptual analysis throughout history. This subject is significant because a metaphysical inquiry into a concept legitimizes both the foundations and the results of the scientific discipline that employs it, and the various ways in which the concept is conceived determine the structure and conceptual boundaries of the discipline.

In this study, a very brief summary of the discussion related to this topic will be provided to highlight the significance of questioning the concept of number in modern mathematical philosophy and to outline the main points of the discussion, within the framework of the reasons stated. However, this summary will not directly focus on the content of the subject in modern and contemporary mathematics; rather, it will emphasize the historical issues to be discussed in this article. Subsequently, ʿAlī al-Qūshjī's definition of number will be examined in terms of its sources and the influence it had, with an effort to identify its historical context. Finally, it will be briefly noted how the question "What is number?" has been addressed both during modern times and in ʿAlī al-Qūshjī's period, along with the possible impacts of the answers given on the mathematical studies of their respective eras. The aim is to lay preparatory groundwork for future research in the history of Islamic philosophy and science on such topics.

Mathematician Karl Weierstrass (d. 1897), at the beginning of a lecture he gave on May 6, 1878, defined number as "a multitude composed of units,"[1] drawing inspiration from Aristotle and Euclid.[2] A student present in the audience who would later become an important philosopher, Edmund Husserl (d. 1938), found this definition insufficient and pursued the question "What is number?" throughout his philosophical life. According to Husserl, number lies at the foundation of the "universal arithmetic" (*arithmetica universalis*) system; therefore, any mathematical philosophy should begin by analyzing the concept of number.[3]

1 The expression of this definition in English sources varies: "The multitude made up of units; multitude consisting of units; multitude composed of units."

2 J. Philip Miller, *Numbers in Presence and Absence: A Study of Husserl's Philosophy of Mathematics* (Dordrecht: Springer, 1982), 31, 41 (footnote 2). In our study, this work has been used as the main source for Husserl's views on the subject.

3 Miller, *Numbers*, 11.

As can be observed even in this brief depiction of Weierstrass and Husserl, the question "What is number?" occupies a central place in modern and contemporary mathematical philosophy; it is also an ancient and profoundly challenging question.[4] On the other hand, as Husserl stated, merely defining number is not sufficient. It is also crucial to understand the concepts used in the definition. Otherwise, the intended definition may become lost in the ambiguity of the new concepts employed in the work of defining. Therefore, it is not enough to merely define number; it is also necessary to observe and even determine how the defined number presents or manifests itself. Manners of presentation relate to many questions, from the type of mathematics we practice to how numbers are utilized in interpreting the world.[5]

I. A Brief Look at the Roots of the Question and Its Modern Development

The definition of number that Weierstrass continued to use at the end of the nineteenth century, which Husserl deemed insufficient as shown in our previous work,[6] can, as reported by Iamblichus (d. ca. 325),[7] be traced back to ancient Egyptian additive numeral systems. It finds its roots in Aristotle's (d. 322 BC) acceptance and detailed examination of the concept, and is fundamentally present in Euclid's (d. ca. 265 BC) mathematical works.[8] This definition was echoed by the Pythagorean

4 As an example, see Penelope Maddy, under the title "What numbers could not be" in the third chapter of her work, where she lists the answers to the question "what is number?" starting with Cantor and Dedekind, along with her critiques and proposals. Maddy, *Realism in Mathematics* (Oxford: Oxford University Press, 1990), 81–106.

5 The answer to the question "What is a number?" is actually the answer to the question "What are natural numbers?" Because, as Leopold Kronecker (d. 1891) said, "Die ganzen Zahlen hat Gott gemacht, alles andere ist Menschenwerk" (God made the natural numbers, all the rest is the work of man). Howard Eves, *Foundations and Fundamental Concepts of Mathematics*, 3rd ed. (Mineola: Dover Publications, 1997), 201; Eric Temple Bell, *The Development of Mathematics*, 2nd ed. (Mineola: Dover Publications, 1992), 170.

6 İhsan Fazlıoğlu, "Aristoteles'in Sayı Tanımı," in *Aded ile Mikdâr: İslam-Türk Felsefe-Bilim Tarihi'nin Mathemata Ma-cerası, 1: 13–27* (Istanbul: Ketebe, 2020).

7 Thomas S. Heath, *A History of Greek Mathematics*, rev. ed. (Mineola: Dover Publications, 1981), 1: 69-70; Heath, trans., *Euclid: The Thirteen Books of the Elements*, 2nd ed. (Mineola: Dover Publications, 1956, 2: 180 (Definition 2). For the views of Hellenistic period thinkers on "number," especially Iamblichus, see Dominic J. O'Meara, *Pythagoras Revived: Mathematics and Philosophy in Late Antiquity* (Oxford: Clarendon Press, 1989).

8 Fazlıoğlu, "Aristoteles'in Sayı Tanımı"; Heath, *Euclid*, II, Book VII, Definitions 2: "To ek monadon synkeimenon plethos."

Nicomachus (d. ca. 120 BC) in his concept of *arithmos*.[9] This development of the discussion, enriched in Islamic civilization and Medieval Europe, partially blurs strict boundaries and dilutes rigid contents; it culminated in the emergence of various understandings of quantity. These developments were shaped by the works of the Italian Bologna algebra school, the inventions of analytic geometry, and advancements in integral and differential calculus, all of which contributed to the evolution of mathematics. Although Immanuel Kant proposed a new perspective on the ontology of mathematical objects within his philosophical system with reference to number (*arithmos*) and magnitude (*megethos*), the historical definition of number mentioned above remained the sole and unmatched definition until the publication of Gottlob Frege's (d. 1925) *The Foundations of Arithmetic: A Logical-Mathematical Investigation of the Concept of Number* in 1884.[10] Frege, opposing this definition, provided a new definition using the concepts of one-to-one correspondence and equality. According to him, "the number belonging to the concept F is a continuation/extension of the concept 'equal to the concept F.'"[11]

Frege emphasized two points in his rejection of the Euclidean definition of number: First, such a definition is only valid for number series starting from two, thus excluding one and zero.[12] Second, the same multitude or addition can be represented by different numbers, leading to ambiguity in the concept of multitude.[13]

According to Husserl, number is, in a sense, a multitude, because every number refers to a certain kind of multitude. However, being a multitude does not mean, as

9 Nicomachus, *The Introduction to the Arithmetic*, trans. Martin Luther D'Ooge (1926; repr. in *Great Books of the Western World*, 2nd ed., Chicago: Encyclopaedia Britannica, 1990) 10: ii: 6.3, 7.3. For the Arabic translation made in the Classical period, see *Kitāb al-Madkhal ilā ʿilm al-ʿadad*, trans. Thābit b. Qurra, ed. Wilhelm Quṭūs al-Yasūʿī (Beirut: n.p., n.d.).

10 Gottlob Frege, *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number*, trans. J. L. Austin (Evanston: Northwestern University Press, 1968). For the Turkish version, see *Aritmetiğin Temelleri*, trans. H. Bülent Gözkan (Istanbul: Yapı Kredi, 2008). In addition to Frege's own work, see also Michael D. Resnik, *Frege and the Philosophy of Mathematics* (Ithaca: Cornell University Press, 1980), particularly the "Arithmetic" subsection of the fifth chapter, titled "Frege's Philosophy of Mathematics," 185–211; William Demopoulos (ed.), *Frege's Philosophy of Mathematics* (Cambridge, MA: Harvard University Press, 1995, repr., 1997), particularly the 6th, 7th, 10th, 13th, and 14th articles.

11 Frege, *Foundations*, 79e. ("The number which belongs to the concept F is the extension of the concept 'equal to the concept F'.")

12 Frege, *Foundations*, 38e.

13 Frege, *Foundations*, 59e.

Euclid asserted, that it is composed of units because multitude = units. What separates a number from any multitude is the existence of a clearly defined quantity (i.e., how many).[14] Thus, a number is not merely an ordinary multitude but rather the modes of "how many."[15] In other words, every number indicates how tightly and specifically a multitude is constrained. Therefore, number is defined as "a determined —limited— multitude."[16] In this way, Husserl revises and adjusts the classical Euclidean definition of number.[17]

If we compare the views of both philosophers summarized above, the following picture emerges: above all, the concepts of determinacy and boundedness bring Husserl closer to Frege. In both definitions, the relational characteristic[18] of the concept of number is emphasized; in other words, number confines and defines multitudes in relation to each other. Indeed, Frege stresses this point by expressing the equality of the two concepts; however, he does not view multitude as a higher genus of numbers; for him, the emphasis on number[19] is an emphasis on the concept, not on multitude.[20] Thus, Frege separates the concepts of "number" and "multitude," thereby eliminating the higher genus. In other words, while Frege outright rejects the concept of *arithmos*, Husserl revises its definition and excludes certain extensions of the Euclidean concept of *arithmos*.

Considering Jacop Klein's definition that "*arithmos*, in any case, means specific numbers of specific things and emphasizes that there are many countable things,"[21] what Husserl does is to make the Euclidean definition clearer by uncovering the meanings that exist within the concept of *arithmos*. Philosophers-scholars like Eu-

14 "The presence of 'a precisely determinate how many'."

15 "modes of how-many."

16 "determinate multitudes."

17 According to Husserl, Weierstrass also arrived at a similar definition in a seminar he gave on October 25, 1880. However, Weierstrass understood the definition as the determination/limitation of the counted objects, rather than the determination/limitation itself. Husserl, on the other hand, means the determination/limitation of "how many" or "how much," not in the sense of the concept under which the individually counted objects fall. See Miller, *Numbers*, 41 (footnote 5).

18 "relational character."

19 "assertion."

20 Frege, *Foundations*, 59e.

21 Jacob Klein, *Greek Mathematical Thought and The Origin of Algebra* (Cambridge, MA: The MIT Press, 1976, repr., Mineola: Dover Publications, 1992), particularly the sixth chapter, titled "The concept of arithmos," 46–60.

doxus, Aristotle, and Nicomachus, who proposed the concept of a defined/bounded multitude, had already found the concept of multitude insufficient during the Hellenistic period.[22]

Husserl responds to Frege's criticisms regarding "0" and "1" as follows: 0 and 1are not numbers in the sense of *anzahlen*; they are, however, numbers in the sense of *zahl*, just like -1 and $\sqrt{2}$.[23] As for the ambiguity in multitude or sum, according to Husserl, there is a difference between "multitude" and "a pile" (or "set"), because only within a multitude (or collection) can each object be taken as one. On the other hand, no object can represent or determine the total by itself; rather, the same object can represent or express itself in different sums. Thus, a number is only determined, limited, and defined when the sum, that is, the multitude, is established.[24]

Now that we have presented this brief context for the concepts of unity and multitude in our ancient mathematical heritage, the definition of number given by ʿAlī al-Qūshjī can be examined.

II. ʿAlī al-Qūshjī: What is Number?

The question "What is number?" as it emerged within the history of philosophy-science along the Seljuk-Ottoman line demands answers on a wide spectrum. For theological inquiries into the multitude emerging from unity, the mathematical accumulation of ancient Greek culture and the responses provided by Neoplatonism to monotheistic religions, particularly Christianity, utilized all available means. Therefore, in all philosophical works produced by Peripatetic philosophers (*Mashshāʾiyya*), the concept of number was to some extent contained in metaphysical inquiries revolv-

22 Heath, *Euclid*, 2: 280 (Definition 2).

23 In German, two words are used for number: *Zahl* and *Anzahl*. *Zahl* refers to real numbers; *Anzahl* is used in everyday language for groups that can be counted, the counted ones; in this sense, *Anzahl* is close to the concept of *arithmos* in Greek. *Anzahl* is sometimes used instead of "cardinal number," while *Zahl* is simply used to mean "number." In Husserl's writing, *Anzahl* is a special case because the essence (meaning) of number is *Anzahl*. This is because, in a mathematical sense, number only emerges after certain operations on *Anzahl*. Miller, *Numbers*, 33, 42 (footnote 11).

24 According to Frege, mathematicians do not deal with the content and meaning of the concept of number in principle; they are more concerned with its relationship to the thing itself, its reference. Husserl, on the other hand, states that the definition should provide the content and emphasizes the distinction between number and the representation of number in the definition of "what is a number?"

ing around the concepts of "unity," "one," "multitude," and "many," along with continuous (*muttaṣil/handasī/miqdār*) and discrete (*munfaṣil/hiṣābī/ʿadād*) quantities and related issues questioned in the context of the category of quantity. On the other hand, an understanding of number that reflects the mathematical implications of natural philosophy based on individual substance (*jawhar-i fard*), produced different results with respect to quantity. This latter approach was developed by *mutakallim*s who fundamentally opposed the Peripatetic philosophers at the level of principles and thus developed a different philosophy-science system. In this study, neither the Peripatetic and *mutakallim* views of number in the context of the quantity problem will be considered; instead, the widely accepted definitions found in mathematics textbooks will be taken into account.[25]

Within the framework articulated above, ʿAlī al-Qūshjī's significant work *Sharḥ al-Tajrīd*, in which he thoroughly addresses quantity and related subjects, as well as the thoughts on the subject he expressed in other works, will not be examined in detail here; only the definition of number in his work *al-Risāla al-Muḥammadiyya fī al-ḥisāb* will be analyzed. References will be made to his *Sharḥ al-Tajrīd* where the ideas are comparable.

1. *al-Risāla al-Muḥammadiyya fī al-ḥisāb*

Abū al-Qāsim ʻAlāʾ al-Dīn ʻAlī b. Muḥammad al-Qūshjī al-Samarqandī (d. 879/1474) was an important member of the Samarqand mathematical-astronomical school, living in the regions of the Timurids and the Aq Qoyunlu. He came to Istanbul and worked until his death at the Sahn-i Semān and Ayasofya madrasas; he wrote numerous books and trained many students. In particular, one should note his work *Sharḥ al-Tajrīd,* in which he expressed his views on metaphysics, epistemology, and the phi-

25 For the philosophy of mathematics in Islamic civilization in general and the concept of number in particular, see Mohammad Saleh Zarepour, "Arabic and Islamic Philosophy of Mathematics," in *The Stanford Encyclopedia of Philosophy* (*Summer 2022 Edition*)*, ed. Edward N. Zalta, last modified April 9, 2022*, https://plato.stanford.edu/entries/arabic-islamic-phil-math/. Also see Mohammad Ardeshir, "Ibn Sīnā's Philosophy of Mathematics", in *The Unity of Science in the Arabic Tradition: Science, Logic, Epistemology and their Interactions*, ed. Shahid Rahman, Tony Street, and Hassan Tahiri (Dordrecht: Springer, 2008), 39–53; İhsan Fazlıoğlu, "Between Reality and Mentality: Fifteenth Century Mathematics and Natural Philosophy Reconsidered," *Nazariyat* 1, no. 1 (2014): 1–39; Hassan Tahiri, *Mathematics and the Mind: An Introduction into Ibn Sīnā's Theory of Knowledge* (Cham: Springer, 2016).

losophy of nature; he applied his ideas presented in this work to astronomy in his *Risālat al-Fatḥiyya fī ʿilm al-hay'a* and to mathematics in his *Risālat al-Muḥammadiyya fī ʿilm al-ḥisāb* (composed in January 1473).[26] Generally speaking, ʿAlī al-Qūshjī aimed to purify the mathematical sciences from Hermetic-Pythagorean mysticism's influence and from the Aristotelian physical and metaphysical principles present in astronomy and optics. He left a lasting impact on the Ottoman-Turkish intellectual and scientific traditions. Additionally, he had significant influences on scientific developments in Turkistan, Iran, India, and Western Europe across various fields.[27]

Up until Bahā' al-Dīn al-ʿĀmilī›s (d. 1030/1621) *Khulāṣat al-ḥisāb*, al-Qūshjī's Arabic mathematical work *al-Risāla al-Muḥammadiyya fī ʿilm al-ḥisāb,* which focused on Indian arithmetic (*ḥisāb-i hindī*), was circulated as an intermediate mathematics book within the Ottoman scientific community. According to Taşköprülüzāde, ʿAlī al-Qūshjī composed and revised it during his second visit to Istanbul in the month of Ramaḍān 877 (January 1473) and presented it to the Ottoman Sultan Meḥmed II. However, according to Kātib Çelebī, ʿAlī al-Qūshjī wrote the work on his way to Istanbul as an envoy of Uzun Hasan and presented it to Sultan Meḥmed II.[28] Upon reading the preface of the work, it can be confidently stated that Taşköprülüzāde's account is correct. The basis of the work's account of Indian arithmetic lies in ʿAlī al-Qūshjī's Persian work entitled *Risālah dar ʿilm al-ḥisāb*, which he wrote in Samarkand.[29] However, Kātib Çelebī, in his commentary *Aḥsan al-Hadiyya bi-Sharḥ al-Muḥammadiyya*, which he composed until the end of the preface to al-Qūshjī's *Muḥammadiyya,* states that the work contains the essence of Ibn Khawwām's *al-Fawā'id al-Bahā'iyya fī al-Qawā'id al-Ḥisābiyya* and Jamshīd al-Kāshī's *Miftāḥ al-ḥussāb*.[30] He also indicates

26 Suleymaniye Manuscript Library, Ayasofya, nr. 2733/2, *taʿlīq*, folios 71b-168b, 11 lines.

27 For detailed information, see İhsan Fazlıoğlu, "Ali Qushjî," in *Dictionary of Scientific Biography, rev. ed., ed. Noretta Koertge* (Detroit: Charles Scribner's Sons, 2008), 1: 45–47. Also see Hasan Umut, "*Theoretical Astronomy in the Early Modern Ottoman Empire:* ʿAlī *al-Qūshjī's Al-Risāla al-Fatḥiyya*" (PhD diss., McGill University, 2019).

28 Kātib Chalabī, *Kashf al-ẓunūn ʿan asāmī al-kutub wa-l-funūn* (*KZ*), ed. Şerefettin Yaltkaya and Kilisłli Rifat Bilge (Ankara: Milli Eğitim Bakanlığı, 1941-1943) 1: 889; Kātib Chalabī, *Aḥsan al-hadiyya bi-sharḥ al-Muḥammadiyya*, Kemankeş nr. 362/4, folio 4a.

29 Ayasofya, nr. 2733/3, with colophon, folios 170b-221a, transcribed by ʿAlī al-Qūshjī in the middle of the month of Ramadan in 877. For a study on it, see Zehra Bilgin, "*Hesab Bilimine Giriş: Ali Kuşçu'nun Risâle der* İlm-i *Hisâb Adlı Eseri - Tenkitli Metin, Çeviri, Değerlendirme"* (*PhD diss.,* Istanbul Medeniyet University, 2021).

30 *Aḥsan*, folio 2a.

that ʿAlī al-Qūshjī's work is a concise summary of the latter,[31] which is correct as far as the subject of the article mentioned below is concerned.

ʿAlī al-Qūshjī clearly states in the introduction of the *Muḥammadiyya* that he wrote the work quickly and in a summarized form to present to Sultan Meḥmed II. However, he later considered writing a more advanced (*mabsūṭ*) work when he found the time.[32] His grandson Mīrīm Çelebī, in his *Sharh al-Fatḥiyya,* and his student Gulām Sinān, in his book entitled *Fath al-Fatḥiyya,* stated that they would write commentaries on the *Muḥammadiyya*, but no copies of these two commentaries exist today. Kātib Çelebī, who provides the same information, points out that these commentaries did not exist in his time, either, labeling Mīrim Chalabī's and Gulām Sinān's statements as "unfulfilled words."[33] Only Kātib Çelebī has commented on the *Muḥammadiyya*, in a work entitled *Aḥsan al-Hadiyya bi-Sharḥ al-Risāla al-Muḥammadiyya*.[34] The *Muḥammadiyya*, of which about twenty copies survive to this day, has been the subject of many contemporary studies.[35]

31 *Aḥsan*, folio 7a.

32 *al-Risāla al-Muḥammadiyya fī ʿilm al-ḥisāb,* Ayasofya nr. 2733/2, folio 74b.

33 *Aḥsan*, folio 2a.

34 İhsan Fazlıoğlu, "Ali Kuşçu'nun *el-Risâlet el-Muhammediyye fi el-hisâb* adlı eserine Kâtip Çelebî'nin Yazdığı Şerh: *Ahsen el-hediyye bi-şerh el-Muhammediyye,*" in *Festschrift in Honor of András J. E. Bodrogligeti*, ed. Kurtuluş Öztopçu, *Türk Dilleri Araştırmaları* 17 (2007): 113–25.

35 *al-Muḥammediyya* was translated into Russian by Ulugbek Atayev in 1972; it has been studied by Gadoyboy Sobirovich, G. P. Matviyewskaya, and H. Tllashev. Boris A. Rosenfeld and Ekmeleddin İhsanoğlu, *Mathematicians, Astronomers, and Other Scholars of Islamic Civilisation and their Works* (*7th-19th c.*) (Istanbul: IRCICA, 2003), 286. The work, cited by many historians of science such as Salih Zeki and Adnan Adıvar, was also extensively introduced by Remzi Demir-Yavuz Unat in "Ali Kuşçu ve *el-Muhammediyye*, *el-Fethiyye* ve *Risâle fî hall eşkâl el-mu'addil li'l-Mesîr* adlı eserlerinin Türk bilim tarihindeki yeri," *Düşünen Siyaset,* no. 16 (2002): 231–55; the *Ḥisāb al-khaṭàayn* (double error calculation) and *taḥlīl* section was published by İhsan Fazlıoğlu as a critical edition, translation, and evaluation from the perspective of the history of mathematics in "Ali Kuşçu'nun el-Muhammediyye fî el-hisâb'ının 'Çift Yanlış' ile 'Tahlîl' Hesabı Bölümü," *Kutadgubilig Felsefe-Bilim Araştırmaları,* no. 4 (October 2003): 135–55. For *al-Muḥammadiyya*, also see Taşköprülüzāde, *al-Shaqā'iq al-Nuʿmāniyya fī ʿUlamāʾ al-Dawla al-ʿUthmāniyya,* ed. Ahmed Subhi Furat (Istanbul: İstanbul Üniversitesi Edebiyat Fakültesi Yayınları, 1985), 160; Kātib Chalabī, *Kashf al-ẓunūn ʿan asāmī al-kutub wa-l-funūn* (*KZ*), 1: 889 and *Aḥsan al-hadiyya*; Salih Zeki, *Āsār-ı Bāqiye*, Istanbul, 1329, vol. I, pp. 195-199; C. Brockelmann, *Geschichte der Arabischen Litteratur* (GAL) (Leiden: Brill, 1938), 2: 235, supplement, 2: 329–30; Süheyl Ünver, *Astronom Ali Kuşçu, Hayatı ve Eserleri* (Istanbul: Istanbul Üniversitesi Fen Fakültesi Yayınları, 1948); Ramazan Şeşen and Cevat İzgi, *Osmanlı Matematik Literatürü Tarihi* (OMALT), ed. Ekmeleddin İhsanoğlu (Istanbul: IRCICA, 1999), 1: 20-27 (nr. 3); İhsan Fazlıoğlu, "Ali Kuşçu," *Yaşamları ve Yapıtlarıyla Osmanlılar Ansiklopedisi* (Istanbul: Yapı Kredi Yayınları, 1999), 1: 216–19; Cevat İzgi, *Osmanlı Medreselerinde İlim* (Istanbul: Küre Yayınları, 2019), 207–8.

2. Definition of Number in the *Muḥammadiyya*

In the *Muḥammadiyya*, after defining the science of arithmetic in the introduction, ʿAlī al-Qūshjī provides the following definition of number:

الحساب: هو العلم بقوانين استخراج مجهولات العددية من معلومات مخصوصة؛ فموضوعه العدد: أعني ما يدخل تحت العدّ ليشمل الواحد، وما يتألف منه. [36]

Arithmetic: It is a science that teaches the rules of inference for deriving numerical unknowns from known quantities with specific properties; its subject is number; by number, I mean everything that falls under the act of counting, including one and everything that consists of one.

ʿAlī al-Qūshjī regards numerals as mere symbols, invented to abbreviate numbers and to facilitate their understanding.

اعلم أن حكماء الهند أرادوا إختصارا في كتابة الأعداد وتسهيلاً للضبط، فوضعوا تسعة أرقام.[37]

"It is known that Indian philosophers agreed on nine digits to abbreviate the writing of numbers and make them easier to understand."

Although ʿAlī al-Qūshjī accepts one as a number, contrary to the classical tradition, he regards zero as a mere symbol— a circle placed to avoid confusion when no numbers occupy a digit. In this context, zero represents the absolute absence of a number and only gains value in a specific digit to indicate the lack of a number.

وكل مرتبة لا يكون فيهاعدد يوضع فيها صفر على صورة دائرة صغيرة لئلا يقع خلل في خفظ المراتب.[38]

"In every digit where a number is absent, a circle-shaped zero is placed to avoid confusion in identifying the digits."

36 *al-Risāla al-Muḥammadiyya fī ʿilm al-ḥisāb,* folio 75a.

37 *al-Risāla al-Muḥammadiyya fī ʿilm al-ḥisāb,* folio 75b.

38 *al-Risāla al-Muḥammadiyya fī ʿilm al-ḥisāb,* folio 76b.

3. *Sharḥ al-Tajrīd*: Unity and Number

To better understand ʿAlī al-Qūshjī's definition of number, it is necessary to briefly consider his thoughts on the concepts of "quantity," "one," "unity," and "multitude." He wrote a commentary known as *al-Sharḥ al-Jadīd* on *Tajrīd al-iʿtiqād* (*al-aqāʾid*) by Nasīr al-Dīn Tūsī, who was also a mathematician-astronomer and whose work is considered a foundational text of Avicennian philosophical theology.[39] It would be natural for them to share similar thoughts on mathematical topics. However, as demonstrated in other studies, ʿAlī al-Qūshjī differs from both the Peripatetics and the theologians in terms of his understanding of both objects (ontology) and knowledge (epistemology).[40] For example, he does not accept individual substance like the Peripatetics, nor does he accept individual essence like the *mutakallim*s. Instead, he supports the concept of the "simple body" attributed to Plato by Fahkr al-Dīn al-Rāzī and embraced by al-Suhrawardī. Without identifying ʿAlī al-Qūshjī's thoughts on all these topics, it is not possible to make definitive statements about his final views.

ʿAlī al-Qūshjī's *Sharḥ al-Tajrīd* should be read alongside Nasīr al-Dīn al-Ṭūsī's text, Shams al-Dīn al-Samarqandī's commentary, and Sayyid Sharīf al-Jurjānī's super-commentary[41]; this strategy is the only way to identify the changes in the philosophical attitudes regarding the issues discussed before his intervention. These contextualizing works belong to the sub-section of "the Regeneration Period" known as "integration of methods," when many philosopher-scientists approached issues through a *hybrid reading* aligned with their preferences. Therefore, in interpreting these texts it is essential to consider, among other factors, the ideas they adopted, and the concepts they omitted, added, or about which they chose to remain silent.

In ʿAlī al-Qūshjī's case, since the text is a commentary, it is not always clear whether the thoughts expressed are his own or if he wrote merely to explain or interpret al-Ṭūsī's perspective. For example, when criticizing or rejecting a particular thought, we must ask: is this critique or rejection made within the philosophical framework of the text itself, or is it from the perspective of other philosophical attitudes? Is

39 ʿAlī al-Qūshjī, *Sharḥ Tajrīd al-ʿaqāʾid*, ed. Muḥammad Ḥusayn al-Zirāʿī al-Riḍāʾī, vols. 1–4 (Qom, 1393).

40 For example, see F. Jamil Ragep, "Freeing Astronomy from Philosophy: An Aspect of Islamic Influence on Science," *Osiris* 16 (2001): 49–71.

41 *Maḥmūd b.* ʿAbd *al-Raḥmān al-Iṣfahānī, Tasdīd al-Qawāʿid fī Sharḥ Tajrīd al-ʿaqāʾid; Ḥāshiyat al-Tajrīd; Sayyid Sharīf al-Jurjānī; Minhuwāt al-Jurjānī, ed. Eşref Altaş, Muhammed Ali Koca, Salih Günaydın, and Muhammed Yetim, vols. 1–3* (*Istanbul, 2020*).

the text enriched by following all its implications to the end? In other words, do ʿAlī al-Qūshjī's explanations represent the text on which he is commenting or his own subjective preferences? The necessity of these questions is not only a contemporary observation, but one frequently noted in the tradition. For instance, Quṭb al-Dīn al-Shīrāzī, in his work *Faʿaltu fa-lā-talum*, states that when responding to the self-critic Muḥammad al-Ḥimādhī, it is necessary to consider different copies of a work and to distinguish between the ideas explained, criticized, and defended in the work by referring to the studies of Aristotle and Galen.[42]

In this context, to understand ʿAlī al-Qūshjī's definition of number, it is essential to follow his position within the entirety of the system step by step: including existence, essence and its attributes, considerations of essence, the simple and compound division of essence, the rulings of parts, individuation, unity and multiplicity, the correspondence (*taqābul*) of unity and multiplicity, the rulings of unity and multiplicity, and correspondence. Within this framework, ʿAlī al-Qūshjī discusses the concept of number under the rulings (*aḥkām*) of unity and multiplicity. Therefore, the most important concepts to pay attention to in this process are "essence," "realization," "determination," "individuation," as well as "consideration" and "secondary intelligibles" referred to as *al-maʿqūlāt al-thāniyya*.[43] Alongside these, the dimensions of unity and multiplicity, the notion of being one and its types must also be taken into account.

Noting this precaution, it is essential to emphasize the following point: ʿAlī al-Qūshjī, agreeing with Nasīr al-Dīn al-Ṭūsī's prioritization of the category of quantity (*kammiyya*) over other categories, justifies this position by stating that the category of quantity is more general (*aʿamm*) than the category of quality and is also more substantial (*aṣaḥḥ*) than other categories in terms of existence. This is because numbers, which belong to quantity, can apply to both material entities (*māddiyyāt*) and abstract entities (*mujarradāt*) that are considered free from qualities. Therefore, quantity is more all-encompassing than quality in terms of existence. In short, a number can be attributed to all categories, even to itself, while quality cannot be attributed to itself. Regarding the claim that it is more substantial than other categories, the other categories are relative accidents.[44] In summary,

42 Qutb al-Dîn Shîrâzî, *Faaltu fa-lâ talum*, Kitâbhâne-i Meclis-i Şurâ-yi Milli, nr.1302, folio 13a.

43 *Sharḥ al-Tajrīd*, v. 1, p. 492.

44 For detailed information, see *Sharḥ al-Tajrīd*, vol. 2, p. 265.

وقد يقال: إنّ العدد يعرض لجميع المقولات حتّى لنفسه. [45]

"It can be said that a number can be attributed to all categories, even to itself..."

ʿAlī al-Qūshjī follows the tradition that embraces the concepts of "unity" and "plurality" as conventional and secondary intelligibles (*al-maʿqūlāt al-thāniyya*). Therefore, he distinguishes between unity and number, and consequently between unity and the number one, using expressions from the *Muḥammadiyya*:

لأنّ العدد – لكونه العدد – يقبل القسمة، والوحدة لا تقبله. ومن جعلها عددا أراد بالعدد ما يدخل تحت العدّ، فالنزاع لفظي. [46]

Number, qua number, accepts division; unity does not. The person who defines "unity" as a "number" is referring to that which falls under the act of counting; therefore, the discussion is merely verbal.

The acceptance of division by a number stems from its falling under the category of quantity because "quantity, by its essence, accepts division."

إنّ الكمّ هو الذي يقبل لذاته القسمة. [47]

According to ʿAlī al-Qūshjī, unity is a relational concept that is related to every being, whether external or mental.[48] Therefore, it serves as a foundational element of every number. For example, the number six is constructed from six-times-unity, not from combinations like three plus three, four plus two, or five plus one. The most common point (*al-qadr al-mushtarak*) among everything discussed is unity; number is also a member of this "everything."

Hence, unity is a principle that establishes the truth of things. This principle can also be applied to numbers: for instance, unity is the most common point that constitutes the truth of six. However, it should not be forgotten that this unity is conceptual. Thinking this way allows for an independent understanding of each number. For

45 *Sharḥ al-Tajrīd*, v. 2, p. 265.

46 *Sharḥ al-Tajrīd*, v. 1, p. 514.

47 *Sharḥ al-Tajrīd*, v. 2, p. 265.

48 For example, unity in species, resemblance; unity in genus, association; unity in quality, similarity; unity in quantity, equality; unity in position, parallelism; unity in relation, correspondence; unity in extremes, conformity. See *Sharḥ al-Tajrīd*, vol. I, p. 512.

example, when we consider the number ten as ten unities, we can grasp the truth of ten without taking into account the different numbers that comprise it. The transition from one number to another can be expressed with the same logic; for instance, moving from the number two to the number three involves adding a new unity to two unities rather than simply adding one to two. This logic gives each number, composed of unities, a unique essence independent of others, while also allowing for the infinite addition of new unities to any number composed of unities.

Each number composed of unities has different truths, and thus each should be treated as a type/kind. The qualities associated with numbers, such as primeness, rationality, and irrationality, stem not from the essence of the number itself but from its concomitants. Of course, differing concomitants lead to differing necessities. It should be reiterated that a number is an abstract concept formed from conceptual unities (*amr*). It is the judgment of the mind that gathers these unities together to yield a number—in other words, the activity of counting. For example, through the power of judgment, the mind adds one unity to another to obtain two (two unities), and then adds a new unity to two unities to establish three (three unities). In short, whether external or mental, every existent (something that has being) has a form of unity, even if it is conceptual. Therefore, unity accompanies existence, i.e., their corroborations are the same.[49]

In light of our readings so far, it can be said that ʿAlī al-Qūshjī follows the ancient approach by considering "numbers to be made up of unities"; however, we have not found evidence that he characterizes them as "multiplicity." We must remind ourselves that *Sharḥ al-Tajrīd* also provides a substantial historical account of each topic in a problematic and systematic way. Therefore, a careful distinction must be made between ʿAlī al-Qūshjī's own views and those he rephrases; this will be the subject of our later readings. To facilitate a comparison, the views presented in the context of the concepts of numbers in the *Tajrīd*, the Isfahanī *Commentary*, and the Sayyid Sharīf's *Super-commentary* are briefly summarized below.

49 *Sharḥ al-Tajrīd*, v. I, p. 490.

4. The *Tajrīd*, Its Commentary, and Super-Commentary: Unity and Number

The discussions around the concept of number in the *Tajrīd* and its commentaries, focusing on the notions of essence, individuation, unity, consideration (*iʿtibār*), and second intelligibles, can be summarized as follows.[50] A number, as a number—therefore as a unity—does not necessarily require multiplicity, just as essence does not necessitate unity and multiplicity to be essence. However, when individuation is added to essence, it multiplies, as individuation is now an existent. Every existent, whether external or mental, must be individuated. This multiplication does not occur externally; it is purely based on the mind's consideration, as the individuation that becomes actualized and multiplies in the external is not its own, but rather external attributes. Additionally, individuation itself is a mental consideration. In this context, the mind abstracts the particular forms depicted by the faculties of the soul, obtaining universal forms and representing them to itself.

Each of the universals depicted in the mind is a "unity," while the particulars depicted in imagination or in other faculties of the soul are subject to "multiplicity." Something that is associated with "multiplicity" is referred to as "existent," but it is not called "one." It can only be called "one" when considered as a whole. Unity and multiplicity are universal; they can only be comprehended by the intellect. Unity is better known in the mind; multiplicity is better known in imagination; yet both are understood through intellect.[51] Each existent, in a sense, is a unity—albeit an abstract one— and is individuated because every existent is one by being a person. In this framework, unity and multiplicity represent a type of form.[52] Unity is therefore the constitutive cause of multiplicity. In other words, the subject of multiplicity is established through the subjects of unity. That is, the composite multiplicity remains faithful to each part of unity. This produces the "consisting of multiplicity from unities."

50 *Sharḥ al-Tajrīd*, v. 1, pp. 395 – 548. The subject of *Sharḥ al-Tajrīd* is discussed in the first section of its primary goal, on trancendentals (*al-umūr al-ʿāmma*), where "existence and non-existence" are addressed. In the second section, essence and its accidents are examined.

51 *Sharḥ al-Tajrīd*, v. I, p. 490-491.

52 *Sharḥ al-Tajrīd*, v. I, p. 493.

اجتماع الكثرة من الوحدات. 53

In other words, “every many is one from a certain perspective.”

كلّ كثير فهو واحد من جهة مّا.[54]

Or, more generally, “every existing thing (*mawjūd*) has a certain type of unity.”

كلّ موجود له وحدة مّا. [55]

However, “consisting” and “being one” are not simply about “coming together”; rather, multiplicity is the complete agreement, overlap, and inseparable interweaving of unities (*ilti’ām*). In this sense, for example, the truth of two is “two-unity.” Thus, to conceive the essence of multiplicity is to conceive its unities; unity is the constitutive element of multiplicity.

إنّ الكثرة ملتئمة من الوحدات؛ فإنّ حقيقة الاثنين مثلا وحدتان. /.../ وتصوّر كنه الكثرة إنّما هو بتصوّر وحدتها؛ فالوحدة مقومة للكثرة. [56]

Still, “unity can be conceived without considering multiplicity.”

يمكن تعقّل الوحدة بدون تعقّل الكثرة. [57]

Unity is an abstract, indivisible essence; thus, it is not a quantity that can be divided inherently. In contrast to genus, kind, and differential unity, individual unity does not accept division into measurable parts. Therefore, the core concept of individual unity is indivisibility, but although the concept of unity is one in essence, it is many in terms of individuals. In summary, all parts of unity are realized under the concept of “unity as unity.” Within the aforementioned principles, a number is not inherently unity because it can be divided; as stated, unity is continuous, whereas a number is discontinuous. From another perspective, a number can be counted, while unity cannot. In this context, those who say, “a number is that which falls under

53 *Sharḥ al-Tajrīd*, v. I, p. 500.

54 *Sharḥ al-Tajrīd*, v. I, p. 507.

55 *Sharḥ al-Tajrīd*, v. I, p. 495, 517.

56 *Sharḥ al-Tajrīd*, v. I, p. 500.

57 *Sharḥ al-Tajrīd*, v. I, p. 502.

the act of counting" consider unity to be something counted in the act of counting; that is, they mean to say, "unity is both a number in itself and the foundational principle of other numbers." Proponents of this view do not adopt the perspective that "a number is half the sum of two equal qualities" and do not see the number as merely discontinuous (*munfaṣil*). For them, each number consists of unities, and the sum of those unities is that number. However, one might ask: since none of them is the essence of unity, can the essence of unity be conceived while excluding all numbers?

According to another perspective, the number "three" is the "three-unity and the form of threeness," which is the principle of all its characteristics. This form does not affect the essence of six, as the existence of six can be considered without the form of threeness. The same applies to other numbers that are thought to be composed of six. Thus, the number six should be viewed as consisting of: 1) unities; 2) numbers (3+3 or 4+2), and 3) six-unity and the form of six, or the forms of other numbers that compose six (such as three, four, or two). However, unities alone are sufficient to form a number; therefore, the forms of numbers do not need to be included in the essence of numbers. Consequently, numbers are composed of unities, not the other way around. This discussion does not include "one" not being seen as a number due to its foundational principle stemming from Ancient Egypt, or the acceptance of "two" as not a number based on some grammarians' unconventional view of "three" as the first plurality.

Numbers composed of unities are, in all their types, conceptual constructs because unity, in itself, is both consideration and secondary intelligibles. Since numbers are composed of unities, they bear the characteristics of primary intelligibles. Thus, due to their abstract nature, numbers do not exist independently in reality; instead, the intellect makes judgments with numbers based on their truths or their individuals (which refers to existents, such as species like human, horse, or cow). Similarly, the forms that underlie the characteristics of numbers are also abstract concepts.

ʿAlī al-Qūshjī's views on the definition of number, when interpreted in light of other thoughts in his works, can be summarized as follows. Humans gather ʻ"data" from objects through their senses and, through abstraction (*tajarrud*), unify them, thus granting them unity and transforming them into objects for the intellect. Through the intellect's judgment (*ḥukm al-aql*), each type within the act of counting becomes a number made of unities. This type of number can be referred to *as a unity*. At this level, each number, such as 5, 10, 23, or 99, is a fused unity. Humans can represent these unities in the imagination as counted entities (*maʿdudāt*). At this level, unity can be referred to as a number. Therefore, at this level, the number 10 can be expressed as: 9+1, 8+2, 7+3, etc. According to ʿAlī al-Qūshjī, whether in the mind or imagination, every composite entity can exhibit both unity and plurality.[58] From this standpoint, humans can match the *unity as a number* in their imagination with objects in the external world and express each object in terms of its quantitative value using numbers.

Before delving into the background of ʿAlī al-Qūshjī's definition of number, two points should be highlighted. As is known, ʿAlī al-Qūshjī was a member of the Samarkand school of mathematics and astronomy and was aware of the works of Jamshīd al-Kāshī, particularly his book *Miftāḥ al-ḥussāb*. Al-Qūshjī's *Muḥammadiyya* is a good example of his knowledge of Jamshīd al-Kāshī's scholarship, as evidenced below. Jamshīd al-Kāshī was the first to rediscover decimal fractions and apply arithmetic operations, despite their long historical development. Given that Byzantine mathematicians referred to decimal fractions as "Turkish fractions" during the same century, namely in the fifteenth century, it can be said that this discovery had a certain level of prevalence[59]. Thus, it can be assumed that ʿAlī al-Qūshjī was familiar with this new type of number.

More importantly, research has shown that one of the most significant features of the *Muḥammadiyya* is the introduction of the terms *mustbat* (positive) and *manfī* (negative), alongside *zā'id* (added) and *nāqiṣ* (subtracted), terms previously used for quantities in arithmetic and algebra.[60] These terms are still used today in countries where Arabic and Persian are spoken, especially in Central Asia and Azerbaijan. Furthermore, these terms were transmitted to Europe by Byzantine mathematicians and translated into Latin as positive and negative numbers/quantities. We use them even in Turkish today.[61] The source of these terms as seen in ʿAlī al-Qūshjī's work remains unclear. Some researchers have claimed that he might have derived this idea from

58 Ali Qushjî discusses the issue by considering many views (*Sharḥ al-Tajrīd*, v.1, pp. 490-492), and then presents his own opinion (*Sharḥ al-Tajrīd*, v.1, p. 492).

59 Herbert Hunger and Kurt Vogel, *Ein Byzantinisches Rechenbuch des 15. Jahrhunderts* (Wien: Der Österreicchiscen Akademie Der Wissensscchaften, 1963), p. 33. Also seem Zeynep Tuba Oğuz, "Ondalık Kesirlerin Osmanlı Muhasebe Metinleri İçindeki Yeri (15. – 17. Yüzyıl)," *DTCF Dergisi,* 57, no. 1 (2017): 446-92.

60 For example, see *al-Muḥammadiyya*, folio 137a.

61 Rosenfeld and İhsanoğlu, *Mathematicians*, 286.

Chinese mathematics, based on his supposed journey to China as an envoy of Ulugh Beg and the writing of a Chinese travelogue (*Khıtāy-nāmah*).[62] However, this claim is based on incorrect information; the travelogue in question belongs to ʿAlī Akbar Khitāyī, who presented his work to Yavuz Sultan Selim in 1516.[63] Recent studies have shown that similar terms, namely *muthbat* and *manfī*, were used by Ibn Hāʾim in his works entitled *Sharḥ al-Urjūza al-Yāsamīniyya fī al-jabr wa-l-muqābala* and *al-Mumtiʿ fī sharḥ al-muqniʿ*. However, these uses do not provide clear and distinct information about the origin of these two terms.[64]

When the new developments mentioned above regarding quantity in general and number in particular are combined with ʿAlī al-Qūshjī's articulation of different ideas in certain fields, especially astronomy, it suggests that, although he incorporated known thought in his works, he may have also been engaged in exploration. Identifying these explorations requires comprehensive secondary academic research on all his works.

III. Roots: From Maragha and Tabriz to Samarqand

ʿAlī al-Qūshjī does not attribute the definition of numbers he recorded in his work to himself; rather, he is aware that he is providing a known and widespread definition. In this context, two points will be clarified below: First, what is the historical background and development of ʿAlī al-Qūshjī's definition? Second, what has been the influence of this definition after ʿAlī al-Qūshjī? The aim here is not to provide a comprehensive account of the definitions related to numbers in the history of Islamic mathematics, nor to investigate the historical development and impact of all these definitions. Both such an attempt and making detailed conclusions about the topic are quite difficult in a field that has seen very little secondary research. Thus, it would be helpful to briefly outline the contours of the research problem.

At the end of the eighth century and into the ninth, a certain logic was employed in Baghdad to merge the scientific heritage of China, India, Turkestan, Iran, Mesopo-

62 Rosenfeld and İhsanoğlu, *Mathematicians*, 287.

63 Kaveh Louis Hemmat, "A Chinese System for an Ottoman State: The Frontier, the Millennium, and Ming Bureaucracy in Khaṭāyī's Book of China" (PhD diss., University of Chicago, 2014).

64 Elif Baga, "Arithmetical Algebra in the Islamic History of Mathematics and Its Peak in the 9th/15th Century: Ibn al-Hāʾim's *al-Mumtiʿ*," *Nazariyat* 3, no. 2 (April 2017): 96–7.

tamia, Ancient Egypt, Greece, and the Hellenistic period, leading to the emergence of new and distinct philosophical-scientific mindsets. These new mindsets inherited many definitions related to numbers from ancient mathematics. Two definitions of "number" stand out, as they can be found in almost every mathematical text of the era. The first is what Weierstrass articulates as "a multitude composed of units," which, as previously noted, was still in use at the end of the nineteenth century. The second is the definition derived from Nicomachus's *Introduction to Arithmetic*, which states that "a number is half the sum of its two sides."[65] From the end of the eighth century to the early twentieth, these two definitions appear in many mathematical works written in Arabic, Turkish, and Persian, as well as in dictionaries and translations into Latin, Hebrew, and Spanish.

These definitions, found in the works of philosophers such as al-Fārābī and Ibn Sīnā, as well as early mathematicians like al-Khwārizmī and ʿAbd al-Qāhir al-Baghdādī, quickly became common ground for nearly everyone engaged in mathematics in one way or another. For example, one can refer to the definition in *Mafātiḥ al-ʿUlūm* by Abū ʿAbdullāh Muḥammad al-Khwārizmī. After presenting arithmetic as "the science of numbers," al-Khwārizmī offers the following definition:

العَدَد هو الكَثْرة المركّبة من الآحاد فالواحد إذاً ليس بالعدد وإنّما هو رُكن العدد. [66]

> "Number is a multitude composed of units; thus, one is not a number; rather, it is an element/principle of the number."

Of course, since the ninth century, many critical approaches may have emerged regarding such definitions of numbers; however, it can be said that these potential alternative definitions did not dominate the mainstream and thus remained exceptional. Numerous debates occurred around these definitions among accountants (*ḥussāb*), geometers (*muhandisūn*), surveyors (*misāhiyyūn*), and even grammarians (*naḥwiyyūn*), and these debates continued until the twentieth century.[67]

65 Nicomachus, *Introduction*, 192, 1-2; *Kitāb al-madkhal*, 20, 14-17.

66 Abū ʿAbd Allāh Muḥammad b. Aḥmad b. Yūsuf al-Kātib al-Khwārazmī, *Mafātīḥ al-ʿulūm*, ed. Jawdat Fakhr al-Dīn (Beirut, 1991), 170.

67 For example, see Ṭahānawī, *Kashshāf iṣṭilāḥāt al-ʿulūm wa-l-funūn*, ed. Rafīq al-ʿAjam et al. (Beirut, 1996), 1167–1168. For the Turkish translation, see Ṭahānawī, *Bilim ve Sanat Terimleri Ansiklopedisi*, ed. Ömer Türker (Istanbul: Ketebe, 2024), 1: 112–14.

These discussions became so widespread that they even entered commonly used dictionaries. Sayyid Sharīf summarized two different stances in his *Taʿrīfāt*: "Number is a quantity composed of units; one is not a number. However, if number is interpreted as something that has its own degrees/order/rank, it [one] falls under the concept of number."[68] Similarly, the unknown author of the fifteenth-century work *Maqālīd al-ʿulūm* stated, "Unity is what makes it possible to say that each existent is 'one'," and then listed the two well-known definitions of number: "Number is a total composed of units" and "Number is half the sum of two quantities."[69] In the first half of the seventeenth century, Muhammad al-Munāwī, who compiled nearly all the definitional works created through the history of Islamic civilization, defined the act of counting as "the consideration (*iʿtibār*) of multiplicity with each other" and then stated, "Number is a quantity composed of units or something that gains concreteness through counting in itself; therefore, one is not a number because it cannot be counted in itself, as being counted in itself is multiplicity itself." Another interesting point Munāwī mentioned is that grammarians emphasized that "one" is also a number in terms of being the foundational principle of all numbers.[70] The approach that saw number as "half the sum of two quantities" was used in the same way in most contexts, while the other definition was accompanied by conceptual changes that should be carefully considered for correct interpretation; definitions such as "number is a multitude composed of units," "number is a quantity composed of units," "number is a total composed of units," "number is the sum of units," and similar expressions, while similar, represent different philosophical stances.

As mentioned above, these two definitions of numbers inherited from the Greek-Hellenistic period were generally accepted in the mathematics produced in the Islamic world. Al-Khwārizmī, the founder of the decimal positional number system and algebra, provided a different definition of number, perhaps unsurprisingly given that he constructed all calculations in both Indian and Arabic arithmetic, as well as algebra and geometry, based on relationality without appealing to a transcendent or immanent metaphysical principle. Since the original work contain-

68 Sayyid Sharīf al-Jurjānī, *Kitāb al-Taʿrīfāt*, ed. Muḥammad ʿAbd al-Raḥmān al-Marashli (Beirut, 2007), 224.

69 *Keys to the Sciences* (*Maqālīd al-ʿulūm*)*: A Gift for the Muzaffarid Shāh Shujāʿ on the Definitions of Technical Terms*, ed. Gholamreza Dadkhah and Reza Pourjavady (Leiden: Brill, 2020), 172.

70 Muḥammad al-Raʾūf al-Munāwī, *al-Tawqīf ʿalā Muhimmāt al-Taʿārīf*, ed. Muḥammad Riḍwān al-Dāya (Dimashq, 1990), 506.

ing al-Khwārizmī's Indian arithmetic has not survived to our time, it is difficult to make an inference based on Latin translations. However, a recently identified manuscript titled *al-Tuḥfa fī ʿilm al-ḥisāb*, written by a mathematician named Abū Naṣr Muḥammad b. Abū al-Mahāmid al-Kāsānī and presented to Abū al-Muẓaffar Giyāth al-Dīn Ṭuluktamur Bak, who was the governor of Crimea and the Right-Hand Bey of the Golden Horde State during the time of the Uzbek Khan (1313-1340), includes a definition of number attributed to al-Khwārizmī. In this manuscript, the author refers to the work of his teacher, Abū Manṣūr Muḥammad b. Muḥammad al-Kāhustuwānī, also known as Ṣadr al-Dīn al-Farazī, titled *Nisāb al-ḥussāb*. In this work, the he criticizes al-Khwārizmī's definition, finding the common definition more accurate:

سئل **محمد بن محمد الخوارزمي** عن عدد يكون نصفه وربعه عشرة، كم هو؟ فقال: ذلك العدد ثلاثة عشر وثلث، وسمي المجموع عددا؛ ولا يصدق عليه هذا الحد، وهو كان معتمدا عليه في هذا الفن. وقال بعضهم: العدد ما تركب من الواحد –أي اجتمع–؛ وأقله اثنان. وهذا أصح العبارات لأن الحساب اتفقوا على أن الأقل من اثنين ليس بعدد.[71]

It was asked of Muḥammad b. Muḥammad al-Khwārizmī, "What is the number whose half and one-fourth equal ten?" He said: "That number is thirteen and one-third; the total is also referred to as a number." This definition, which al-Khwārizmī relied on in his science of calculation, is not suitable for the [concept of] number. Some also said: "A number is something that is composed of ones; by 'composed' it is meant to be a sum, and the smallest is two." This is the most accurate expression because calculators have agreed that anything smaller than two is not a number.

What is stated above can be explained as follows: Let us assume this equation: $\frac{1}{2}x+\frac{1}{4}x=10$. Reducing to the first of the simple equations, which is $ax=c \Rightarrow x=\frac{c}{a}$ (division) or $x=\frac{1}{a.c}$ (ratio), one yields $3x=40$; thus obtaining $x=13+\frac{1}{3}$.

Al-Khwārizmī directly refers to the total (*majmūʿ*) as a number. In this definition attributed to him, the sensitivity of *arithmos* and *megethos* from ancient mathematics is not evident. There is also an algebraic number; furthermore, a quantity expressing an approximate value can also be referred to as a number. Lastly, outside

71 Abū Naṣr Kāsānī, *al-Tuḥfa fī al-Ḥisāb*, Süleymaniye Library, Ayasofya nr. 2723, folio 3a-3b.

the functional relationships of mathematics, a number does not have an ontological reality. Interestingly, perhaps as a result of this approach, "1" is not considered a number within number theory (*ʿilm al-ʿadad*) in the Islamic context, while in geometry, measurement, and accounting sciences, 1's status is a subject of debate. In algebra (*ʿilm al-jabr wa-l-muqābala*), however, it is accepted as a number.[72]

I have evaluated Abū Naṣr al-Kāsānī's account of the definition of number and his own thoughts in another work; here, it will suffice to note that Abū Naṣr al-Kāsānī adopts the prevailing opinion, except for the definition of number he attributes to al-Khwārizmī through his teacher, with which he disagrees.[73] However, the definition he shares is significant because it is not found in any other historical sources, and it aligns with al-Khwārizmī's general stance. Since we cannot verify whether this attribution to al-Khwārizmī is accurate, the definition may also have been attributed to him by those who wished to benefit from his prominence in the history of mathematics. Whatever the case, identifying the origins of the definition mentioned by ʿAlī al-Qūshjī in his work can only be accomplished by examining the related works throughout the entire history of Islamic philosophy and science, as indicated several times above. This study will focus on the definitions of number expressed above and will largely rely on mathematics books as material.

The investigations conducted thus far suggest that the context in which ʿAlī al-Qūshjī's definition has become widely discussed and included in textbooks is likely the scholarly environments created by the mathematical-astronomical schools of Maragha and Tabriz. Within this framework, ʿAlī al-Qūshjī's definition of a number has philosophical and theological roots in work by Naṣīr al-Dīn al-Ṭūsī, Shams al-Dīn al-Iṣfahānī, and Sayyid Sharīf al-Jurjānī, while its mathematical and calculative roots can be traced to Ibn Khawwām, Kamāl al-Dīn al-Fārisī, Niẓām al-Dīn al-Nīsābūrī, and Jamāl al-Dīn al-Turkistānī, as well as Abū al-Ḥasan Bahmanī and ʿAlī b. Gharbī. Ultimately, these components would also be interpreted by Jamshīd al-Kāshī in Samarqand.

Niẓām al-Dīn Nīsābūrī (d. 1329), a student of Quṭb al-Dīn al-Shīrāzī, is very clear and definitive regarding the definition of a number:

72 For example, one can refer to the "number/*adad*" entry in Ṭahānawī's *Kashshāf* mentioned above.

73 İhsan Fazlıoğlu, "Altın-Orda Ülkesi›nde İlk Matematik Kitabı: Hesap Biliminde Şaheser [*et-Tuhfe fî ilmi'l-hisâb*]," in *Aded ile Mikdâr: İslam-Türk Felsefe-Bilim Tarihi'nin Mathemata Ma-cerası* (*Istanbul: Ketebe, 2020*), 1: 116–20.

الحساب علم يُعْرَف فيه طرق استخراج مجهولات العدديّة من معلومات مخصوصة. فموضوعه العدد، وهو كميّة تطلق على الواحد وعلى ما يتألف منه. / .../ والحكماء اختلفوا في أنّ الواحد هل عدد أم لا. الحقّ أنّه عدد كما ذكرنا. [74]

> Arithmetic is a science that teaches the methods of obtaining the unknown quantities from known specified ones. Its subject is the quantity that is called 'one' and that which is composed of ones, which is called a number. [...] Philosophers have debated whether one is a number; however, as we mentioned, the correct view is that one is indeed a number.

Kamāl al-Dīn al-Fārisī, in his commentary on his teacher Ibn al-Khawwām's *al-Fawāʾid*, examines the definition of the *ḥisāb* by referring to the definition of his teacher, that "*ḥisāb* is the science of known quantities from which unknown quantities are obtained."[75] In this context, al-Fārisī begins by stating that the category of "quantity" is divided into continuous (*muttaṣil*, *miqdār*) and discrete (*munfaṣīl*), with discrete quantities being referred to as "numbers" (*aʿdād*). He points out that numbers are defined as "quantities composed of ones." However, he indicates that such discussions belong to the realm of metaphysics (*al-ʿilm al-aʿlā*). Al-Fārisī emphasizes the necessity of distinguishing between the theoretical aspect, called *ʿilm al-ʿadad*, and the practical aspect, called *al-ḥisāb*, in discussions on numbers.

Here, the term "practical" encompasses not only external calculation processes but also mental calculation processes. He suggests that this distinction will determine the answer to the question "What is a number?" and moves on to Ibn al-Khawwām's definition: "A number is the sum of ones." Al-Fārisī states that this definition was formulated with consideration for the necessities of numbers, rather than for the absoluteness of the number itself. He reformulates Ibn al-Khawwām's definition by stating: "Thus, a number is a quantity that arises from the sum of ones," and he points out that this definition reflects the view of those who consider "one" to be a "true one," indicating that such individuals, like Euclid, do not accept fractions as numbers. However,

74 Niẓām al-Dīn Nīsābūrī, *al-Shamsiyya fī al-Ḥisāb*, ed. Elif Baga (Istanbul: Türkiye Yazma Eserler Kurumu Başkanlığı, 2020), 124–5.

75 İhsan Fazlıoğlu, "İbn el-Havvâm (öl. 724/1324), Eserleri ve el-Fevâid el-Bahâiyye fî el-Kavâid el-Hisâbiyye'deki Çözümsüz Problemler Bahsi," *Osmanlı Bilimi Araştırmaları Dergisi, no. 1* (1995): 69–128, 364–7 (English summary). See also Fazlıoğlu, "İbn el-Havvâm (öl. 724/1324) ve Eseri *el-Fevâid el-Bahâiyye fi el-Kavâid el-Hisâbiyye*: Tenkitli Metin ve Tarihi Değerlendirme" (M.A. thesis, Istanbul University Institute of Social Sciences, 1993).

calculators (*ḥussāb*) do accept fractions as numbers; thus, for them, a number is not merely the sum of ones. For calculators, a number is defined as "a quantity obtained from one through repetition (*taqrīr*) and division (*tajziʾa*), or both."[76]

Another member of the Tabriz mathematics-astronomy school, Jamāl al-Dīn al-Turkistānī, a close associate of highly respected by Quṭb al-Dīn al-Shīrāzī and another teacher of Kamāl al-Dīn al-Fārisī, discusses the definitions of concepts used in the science of arithmetic in the introduction of his work *al-Risāla al-ʿAlāʾiyya fī al-masāʾil al-ḥisābiyya*, which was widely used along the Turan-Iran-Anatolia line. The first concept he addresses is "number," and he provides a very brief definition:

العدد ما يقع في العدّ ويشمَل الواحدَ وأكثرَ منه. وهو إمّا مُطْلَقٌ وإمّا منسوب إلى جملةٍ
تُفْرَضُ واحدةً وهو الكسور.[77]

Number is that which falls under the act of counting; it encompasses one and [all quantitites] greater than one. This number is either absolute[78] or relative to a whole assumed as one, the latter being fractions.

The readers of the copies we use in this study—the III. Ahmad and Laleli manuscripts— perhaps justifiably engage in a discussion with Jamāl al-Dīn al-Turkistānī regarding everything mentioned so far in the context of numbers. They convey to him, and consequently to potential readers, that the author has provided an incorrect definition. In this context, they remind him of both definitions: "A number is half the sum of two equal quantities" and "A number is the sum of ones." In the Laleli manuscript, another reader states: "These definitions contradict the principle by which the author defines the number, because the author's definition encompasses both one and all numbers beyond one," thus cautioning potential readers about the author's intent. This situation demonstrates how the common definitions of numbers became publicized in accordance with the principle of the "orientation of knowledge," which has been the subject of discussion elsewhere.

76 Kamāl al-Dīn Fārisī, *Asās al-qawāʿid fī uṣūl al-fawāʾid*, ed. Muṣṭafā Mawaldī (Cairo, 1994), 68–71.

77 Jamāl al-Dīn al-Turkistānī, *al-Risāla al-ʿAlāʾiyya fī al-masāʾil al-ḥisābiyya*, Süleymaniye Manuscript Library, Laleli no. 2729, folios 1b-2a. Also see Topkapı Museum Manuscript Library, III. Ahmed no. 3669, folio 2a.

78 That is, "one, two, ten, twenty, etc." are not considered in relation to a whole assumed to be "one." Laleli no. 2729, folio 2a, margin.

Jamāl al-Dīn al-Turkistānī is the focus of discussions not only in the copies of his work but also in two commentaries written about it. Abū al-Ḥasan ʿAlī b. Muḥammad b. ʿAlī b. Kaykhusraw al-Bahmanī, who wrote a commentary on the work, interprets this sentence in detail:

(العدد ما يقع في العدّ ويشمَل الواحدَ وأكثر منه.) وقال بعض آخر: هو كميّة تُطْلَق على الواحد وعلى ما يتألف منه. وهذا التعريف إنّما يكون جامعاً عند من يجعل الواحد حقيقياً، ولم يقل بالكسور كإقليدس. وأمّا عند الحُسّاب القائلين بالكسر فلا. قيل: الصواب أن يقال العددُ كميّة تطلق على الواحد وعلى ما يتحصّل منه بالتكرير أو بالتجزئة أو بهما.

وقالت طائفة من أهل التحقيق العدد هو الكثرة المجتمعة عن الوحدات. وقال بعض آخر هو ما كان نصف مجموع حاشيتيه. فلا يشمل الواحد بهذين التعريفين.

واعلم أنّ العدد يبحث عنها بوجهين. الأوّل من حيث أن يثبت لها أو يسلب عنه أعراض ذاتيّة فيكون مسائلها من العلوم النظريّة. وهو جدير بأن يسمى علم العدد. والثاني من حيث أن يُتَوَصَّلَ من معلوماته إلى ما لم يعلم من خواصّه ولوازمه فيبحث عن كيفية ذلك (ذلك التوَصّل) فيكون مسائلها من العلوم العمليّة، وهو علم الحساب. [79]

(Number is that which falls under the act of counting; it encompasses one and [all quantitites] greater than one.) Others say: "It is the quantity given as a name to that which is composed of one and more than one." This definition may be considered comprehensive in the eyes of those who regard one as truly one; they do not consider fractions [as numbers], like Euclid. However, in the view of the accountants (*ḥussāb*), who consider fractions [as numbers], this definition is not comprehensive. It is said that it is more accurate to state: "A number is the quantity that is named for what arises from one and one through repetition or division, or both."

A group of the people of verification has said: "A number is a multitude composed of units." Others have stated that "a number is half the sum of its two sides." These two definitions do not encompass one another.

A number can be investigated from two perspectives. The first is to prove or remove the essential properties of a number; this kind of issue belongs to the theoretical sciences, and it is appropriate to refer to this field as the science of numbers (*ʿilm al-ʿadād*). The second is to derive unknown properties and necessities from known properties (*ḥawāṣṣ*) and necessities (*lawāzim*) of a number. Investigating this quality of derivation makes the study of number a practical science, which is the science of calculation (*ʿilm al-ḥisāb*).

79 Abū al-Ḥasan al-Bahmanī, *Sharḥ Risālat al-ʿAlāʾiyya*, St. Petersburg, no. 1069, fol. 2a. The work, if the manuscript is not incomplete, does not contain the sections on the "calculation of errors" and "the science of algebra and balance" from the *ʿAlāʾiyya*.

The statements of Abū al-Ḥasan al-Bahmanī evoke Kamāl al-Dīn Fārisī's sentences in many ways. Since we do not have information about al-Bahmanī's life story, we cannot say whether he belonged to the Tabriz mathematics-astronomy school. However, his work is significant, at least in terms of demonstrating how widespread the ideas developed around the subject were within the Tabriz mathematics-astronomy school. This spread is particularly relevant because the Behmenids were a dynasty that ruled the Deccan region of India between 1347 and 1527,[80] and it appears that Abū al-Ḥasan al-Bahmanī dedicated his work to Sultan Ghiyāth al-Dīn Abū al-Muẓaffar Muḥammad Khān of this dynasty.[81]

Jamāl al-Dīn Turkistānī's work has a detailed second commentary written by his likely student, Jalāl al-Dīn ʿAlī b. al-Gharbī, titled *al-Muʿjizāt al-Najībiyya fī Sharḥ al-Risāla al-ʿAlāʾiyya*,[82] which al-Gharbī dedicated to Najīb al-Dīn Muḥammad b. Amīr Shams al-Dīn al-Dāmaghānī. In his work, ʿAlī b. al-Gharbī discusses the definitions of number examined in this study, exploring their various extensions through a virtual dialogue method in a question-answer format. He refers to the works of the Tabriz mathematics-astronomy school, particularly those of Ibn Khawwām and Kamal al-Dīn al-Fārisī, as well as of a scholar named Naṣīr al-Dīn al-Kāshī.[83] He conducts this discussion based on the definitions of *ḥadd*/definition and proposition found in the work *al-Shamsiyya fī al-manṭiq* by Najm al-Dīn al-Kātibī, whom he specifically names. We will not cover all of ʿAlī b. al-Gharbī's discussions related to the topic in this work. Instead, we will highlight a few key points: 1. ʿAlī b. al-Gharbī considers the roles and relationships of numbers in operations while addressing the topic. 2. He does not find it appropriate to leave the subject solely to metaphysics. 3. He determines his position based on the acceptance that one is a number and conducts a detailed discussion to support this stance.[84]

Although discussions regarding the definition of number have a long historical trajectory, the historical context in which the subject was comprehensively ad-

80 Enver Konukçu, "Behmenîler," in *Türkiye Diyanet Vakfi İslam Ansiklopedisi* (Istanbul: TDV Yayınları, 1992), 5: 353–4.

81 Abū al-Ḥasan al-Bahmanī, *Sharḥ Risālat al-ʿAlāʾiyya*, folio 1b.

82 ʿAlī b. al-Gharbī, *al-Muʿjizāt al-Nağībiyya fī Sharḥ al-Risālat al-ʿAlāʾiyya*, Topkapı Palace Museum Library, III. Ahmed no. 3117. The manuscript was copied by the author's student on the 8th of Rebiülevvel 773 / 19th September 1371, and was also reviewed under the author's supervision.

83 ʿAlī b. al-Gharbī, *al-Muʿjizāt, folios 5a-11a.*

84 ʿAlī b. al-Gharbī, *al-Muʿjizāt, folios 7a-11a.*

dressed, incorporated into textbooks, and became widespread and publicized was, as we have demonstrated, the Tabriz mathematics-astronomy school and its aftermath. Indeed, if we consider the existing manuscripts written by the aforementioned names, this fact becomes even clearer. It can be said that the definition of number in these works became a widely accepted definition, particularly along the Turan–Iran–Anatolia line. Indeed, prior to ʿAlī al-Qūshjī, Maḥmad Shāh,[85] the son of Shams al-Dīn Fanārī, who reorganized scientific life in the Ottoman realm, treats the definition in his work *Anmūdhaj al-ʿUlūm* as a conventional and widely recognized one: "Number is that which falls under the act of counting; it also encompasses one."[86]

The historical background of the definition of number in ʿAlī al-Qūshjī's work culminates with the developments at the madrasa built by Ulugh Beg in Samarqand. In this madrasa, the famous work *Miftāḥ al-ḥussāb* by Jamshīd al-Kāshī, who was likely both ʿAlī al-Qūshjī's teacher and the first director of the Samarqand Observatory, attained its final form and was incorporated by ʿAlī al-Qūshjī into the *Muḥammadiyya*. This final formulation can be summarized as follows.[87] The subject of the science of calculation, defined as "the rules for obtaining numerical unknowns from knowns," is number. A number is that which falls under the act of counting; it encompasses one and those composed of one. In one respect, it is a substantial quantity (*ṣaḥīḥ*), and, in another, it is a relative quantity (fractions, *kusūr*). Quantity is that which is given as an answer to the question "How many?"

ʿAlī al-Qūshjī placed these discussions and definitions related to numbers in his intermediate-level book, *Muḥammadiyya*, thereby stimulating interest in the subject in Istanbul. This interest led to the widespread adoption and discussion of the defined concept. The specific results of these discussions in number theory and in the science of calculation (arithmetics, algebra, and geometry) are subjects for further research in mathematical sciences.

85 Maḥmad Shāh Fanārī, *Anmūdhaj al-ʿUlūm*, Süleymaniye Manuscript Library, Hüsrev Paşa nr. 482, folio 172b.

86 For detailed information, see İhsan Fazlıoğlu, "*İthâf*'tan *Enmûzec*'e Fetih'ten Önce Osmanlı Ülkeesi'nde Matematik Bilimler," in *Proceedings of the International Molla Fenârî Symposium* (*4-6 December 2009, Bursa*), ed. Tevfik Yücedoğru, Orhan Ş. Koloğlu, U. Murat Kılavuz and Kadir Gömbeyaz, 131–63 (Bursa: Bursa Metropolitan Municipality Publications, 2010).

87 Jamshīd Kāshī, *Miftāḥ al-ḥussāb*, ed. Aḥmad Saʿīd al-Demirdaş and Muḥammad Ḥamdī al-Ḥifnī al-Shaykh (Cairo, n.d.), 44; ed. Nādir al-Nābulusī (Dimashq, 1977), 47.

IV. Effects: Discussions in Istanbul

As noted above, the impact of ʿAlī al-Qūshjī, and thus of the line he represented, on subsequent developments, particularly in the Ottoman Empire, requires long-term research. This study will proceed through two representative examples and will connect the topic with one of the last great figures of the classical tradition, Taqī al-Dīn al-Rāṣid. The two examples in question are the works of Fanārīzāde ʿAlī Çelebī, representing the high mathematical culture of the Ottoman Empire, and Kātib ʿAlāʾ al-Dīn Yūsuf, who reflects the influence of this culture in accounting mathematics.

According to the information provided by the sources, Fanārīzāde traveled to Turkistan and Iran; studied transmitted, philosophical, and mathematical sciences in centers such as Herat, Samarqand, and Bukhara; and taught at the madrasas in these centers for a time. He wrote a commentary on Sirāj al-Dīn al-Sajāwandī's *al-Tajnīs fī al-Ḥisāb* in which he insists on the definition of number in both the arithmetic and algebra sections of this work.[88] He first distinguishes between number theory and accounting science using the theory of the classical sciences (logic, *ʿilm al-manṭiq*). He states that number theory, also called arithmetic, examines the intrinsic properties of numbers—such as being a number, evenness, primality—while accounting science investigates the method for obtaining unknown numbers from known numbers (*istikhrāj*). In this context, he considers number theory to be among the principles of mathematics (*riyāḍiyyāt*), whereas accounting science pertains to applied fields involving specific operations like multiplication, division, and ratios.

After outlining this framework, Fanārīzāde highlights different approaches to the question, "What is a number?" He notes the first group that states, "A number is a quantity composed of one and more than one." This group does not refer to a terminological quantity (*iṣṭilāḥī*); otherwise, they would not call it a number, but would use something that answers the question "How many?" The second group states, "A number is a quantity composed of unities." The author upon whom Fanārīzāde is commenting, al-Sajāwandī, considers a number to be half the sum of two equal quantities, meaning that what is considered a number would not actually be a number according to this view. A portion of the second group that lacks investigative knowledge does not accept two as a number. Fanārīzāde explicitly states, "We follow the first group's line," promising to clarify this position later.[89]

88 Fanārīzāde ʿAlī Chalabī, *Sharḥ al-Tajnīs fī al-ḥisāb*, Topkapı Sarayı Müzesi Kütüphanesi, III. Ahmed nr. 3154.

89 Fanārīzāde ʿAlī Chalabī, *Sharḥ al-Tajnīs fī al-ḥisāb,* folios 1b-2a.

Fanārīzāde does not stop with these explanations in the introduction of his commentary; he returns to the topic in the algebra section to address algebraic quantity. He emphasizes that number, as a discrete quantity (*al-kamm el-munfaṣil*), is "self-sufficient" (*al-qāʾim bi-nafsihī*). Thus, when it is conceived (*taʿaqqul*), its reference to something else (*ghayr*) is disregarded; the self-conceptualization of a number ceases when that otherness is considered. Therefore, when the term "number" is mentioned, all references and operations are excluded. When referred to in another context, it takes on names such as root (*jazr*), square (*mâl*), etc.; when subjected to operations, it acquires names such as multiplication, division, ratio, etc. Fanārīzāde's division can be linked to ʿAlī al-Qūshjī's notion of "number as unity" in the first part, and "unity as number" in the second part.

After brief and useful philosophical explanations, Fanārīzāde summarizes by stating, "Discrete quantity is called a number when considered as such (*min ḥaythu huwa*)." He then revisits the well-known definitions of number. Before examining these definitions, it is important to emphasize that the concept of "consideration" (*iʿtibār*) in the sentence is significant because it connects to ʿAlī al-Qūshjī's view of unity and number as considered concepts. According to these definitions, some scholars state that "a number is something (*mā*) composed of units," meaning it is derived from its similarities or from aggregates of its similarities. Others argue that "A number is a multitude constituted by units." According to Fanārīzāde, the first definition is formulated with something more implicit, while the second requires interpretation for its validity. Al-Sajāwandī found both definitions inadequate and cited his own preferred definition of side (*ḥāshiya*).

To draw the reader's attention, Fanārīzāde begins a sentence with "You also know that..." and states that the definition al- Sajāwandī deemed sufficient is similar to the previous definition. He then emphasizes that the definition is correct according to the second of the three views (*madhhab*) he mentioned in the introduction of the commentary, but not according to the first and third views. According to him, those who investigate the rules of this science and work in depth understand that the majority adopt the first view. Finally, just as he did at the beginning, Fanārīzāde states, "That is why the correct definition is the one that aligns with this view, and in this commentary, we also prefer it."[90]

90 Fanārīzāde ʿAlī Chalabī, *Sharḥ al-Tajnīs fī al-Ḥisāb,* folios 66a-67a.

One of the expressions that stands out in Fanārīzāde's discussion is the phrase "what is given as the answer to the question of 'How many?'" A similar sentence, as previously noted, was also written by Jamshīd al-Kāshī. When the expressions are considered as a whole, it is understood that "quantity," and thus "number as a discontinuous quantity, refers to everything that answers the question of how many." Interestingly, the answer given to the question "What is number?"— stated as "Everything that answers the question of how many is a number"—carries a strong linguistic influence. Indeed, in the most important syntax textbooks of the Ottoman madrasas, *al-Kāfiya fī al-Naḥw,* written by Ibn al-Ḥājib (d. 1249), he cites Ḥasan b. Sharafshāh al-Astarābādī (d. 1315?), who wrote one of the most widely circulated commentaries under the title *al-Wāfiya fī Sharḥ al-Kāfiya* in the Ottoman scholarly public, as saying the following.

أسماء العدد ما وُضِع لكمّيّته آحاد الأشياء – أي أسماء العدد وُضِعت ليدلّ على كمّيّة آحاد الأشياء – أي المعدودات–.

فالواحد والاثنان عدد لوقوعهما جواباً عن القول الفاعل: كم عندك؟.[91]

The names of numbers have been established for the quantities of the unities of things. That is, the names of numbers are designated to indicate the quantity of the unities of objects, that is, the counted (*maʿdūdāt*). One and two are also numbers because they serve as answers to the question 'How many (are there) besides the doer?'

Considering that the Ḥasan b. Sharafshāh al-Astarābādī was a student of Naṣīr al-Dīn al-Ṭūsī and that, following Quṭb al-Dīn al-Shīrāzī's journey to Anatolia, he was held responsible for monitoring the students in Maragha, it can be said that he was well-versed in mathematical topics. The public prevalence of this argument can also be seen in the notes found on the flyleaves and margins of manuscripts. For instance, in the philosophical works of Taşköprülüzāde, housed in the Berlin Stadtbibliothek, Springer 1823, both the concept of unity and the definition of number are provided on the flyleaves: "One is not considered a number by the scholars of philosophy; however, since the scholars of Arabic define 'number' as 'that which falls under the act of counting,' 'one' is also a number in their view." What is interesting here is that Ḥasan b. Sharafshāh al-Astarābādī's work, *al-Wāfiya fī Sharḥ al-Kāfiya*, is cited as a source for this discussion and for the preference of the linguists.[92]

91 Bāyezīd nr. 11101, folio 119b.

92 Berlin Stadtbibliothek, Springer nr. 1823, folio 2b.

It is noteworthy that one of the most important texts that examines and compares these number definitions is found in a counting-mathematics work authored by an Ottoman accountant. Kātib ʿAlāʾ al-Dīn Yūsuf (d. 917/1512), in his work *Murshid al-Muḥāsibīn* written in 917/1512, compares number definitions and discusses the status of one as a number.[93] According to him, based on both Pythagorean definitions of number, it is not possible to consider one as a number. He emphasizes that the most important point in the discussion is the definitions of the terms, as both groups may define the terms differently. In particular, the concepts of "one" and "unity" should not be conflated. In this context, he states that if unity is attributed to an object, it is called "one"; if number is attributed to an object, it is called "counted" (*maʿdūd*). Therefore, unity is an attribute of number. From this perspective, one cannot define number based on what is counted. The act of counting cannot be taken to mean that something is counted. In this framework, as long as one pays attention to the definitions of terms, accepting number as defined by ʿAlī al-Qūshjī is more appropriate.[94]

Lastly, the views of the mathematician-astronomer Taqī al-Dīn al-Rāṣidcan be considered on this matter.[95] Taqī al-Dīn al-Rāṣid provides a new definition for the concept of unity, which is at the root of the issue. According to this definition, "unity" is that which is one with itself; thus, unity is the quality of that thing. He then addresses the second source of the problem, the concept of multiplicity, in a different manner. For him, multiplicity is also an attribute of number. Multiplicity is composed of ones and is referred to as a discrete quantity. Taqī al-Dīn al-Rāṣid's statements can be interpreted as follows. Each number, as a quantity, is a multitude composed of ones. Additionally, because it exists in its ranks (*marātib*) and has other properties beyond those stipulated in Euclid's *Elements*, it is considered a number. Taqī al-Dīn dismisses this definition, responding to, "It is not a number because a number is half the sum of its attributes" with a single word: "Meaningless" (*laysa bi-shayʾ*). Finally, without going into detail, Taqī al-Dīn records an interesting proposition: "The ranks of a number (*marātib*) are conventional; even though the one counting and the counted are limited, there is no limit/end to it."[96]

93 For copies: *OMALT*, v. I, p. 46.

94 *Murshid al-muhāsibīn*, Berlin nr. 2398, author's manuscript. I am grateful to my dear friends Hakan Yıldız and Zülfikar Kam for providing me with the microfilm of this manuscript. I also thank the administration of the Science and Art Foundation for undertaking the process of publishing the microfilm.

95 Taqī al-Dīn Rāṣid, *Bughyat al-Ṭullāb min ʿIlm al-Ḥisāb*, Süleymaniye Kütüphanesi, Carullah nr. 1454.

96 See Appendix 6.

Undoubtedly, the definitions of numbers and their extensions discussed in this research can also be observed in later studies conducted in the Ottoman Empire. Particularly Bahāʾ al-Dīn al-ʿĀmilī's seventeenth-century *Khulāṣat al-Ḥisāb*, which was used as a textbook in Ottoman madrasas along with its commentaries, contains highly detailed information on the topics of *ʿilm al-ḥisâb*, *ʿilm al-ʿadad*, and the concepts of unity, multiplicity, and number. The identification and evaluation of this information will be the subject of our future studies. However, one example can point to an important aspect we initially sensed: particularly toward the end of the second half of the seventeenth century, discussions about numbers and related topics began to be largely relegated to *ʿilm al-ilāhī* (metaphysics); thus, *riyāḍī* sciences, particularly *ʿilm al-ḥisāb*, came to be viewed more as an applied field of "operations." One of the significant negative consequences of this shift is that the primary intellectual stimulus for engaging with number theory in ancient times had been a kind of number theology, or, in other words, a form of number mysticism. All mathematicians dealing with number theory in the pre-modern Ottoman period likewise tended toward a form of number theology, or even number mysticism. For example, the work of Munajjimbāshī Aḥmad Dede, which addresses number theory, exhibits a strong Hermetic-Pythagorean influence and even leans toward Neoplatonic philosophy.[97] Even Fermat, who is regarded as the founder of modern number theory in the early modern period, dealt with number theory through a Pythagorean lens.

Conclusion: Different Problems, Different Purposes

In modern mathematics, the definition of number was primarily a concern to establish the foundation on which calculus is based. This concern was because the infinitesimal or infinitesimal quantities, while present in the foundation of calculus, remained a problem that philosophers and scholars needed to resolve.[98] Berkeley referred to Newton's concept of fluxions, labeling the infinitesimal as "the ghosts of dead quantities." Voltaire, one of the Enlightenment's writers, provided an ironic definition of calculus as "the art of counting and measuring what cannot be comprehended."[99] However, in the nineteenth century, mathematicians began to question

97 *Ghāyat al-ʿudad fī ʿilm al-ʿadad*, Veliyüddin nr. 2329/1, folios 1b-68b.

98 Eves, *Foundations*, pp. 173-9.

99 Boyer, HofM,1968:470; Kline MinWC, p. 232.

the foundations of calculus and sought to provide it with a solid basis. Prominent mathematicians such as Weierstrass, Cauchy, Dedekind, and Cantor grappled with these issues.[100] As a result of these prolonged discussions, the fundamental ideas developed by Weierstrass are still shared today. According to him, "a program should be created in which the real number system will be prioritized, and then all the fundamental concepts of analysis will be derived from this number system."[101] This program was called "the arithmetization of analysis," or, in other words, *aritmetica universalis*, meaning "the analysis of the concept of number and the science based on this analyzed concept."[102] Once this program was successful, the obstacles in front of calculus were removed; differential calculus was derived from the properties of the real number system, and the necessity to rely on ambiguous/indeterminate (and thus mystical) concepts like "infinitesimal quantities" was eliminated. Thus, classical analysis was rigorously reconstructed based on the real number system.[103]

In the classical period, the goal of trying to define number was to escape from Hermetic-Pythagorean mysticism, in other words, from arithmology or the theology of numbers. Logically determining one of the fundamental concepts of human knowledge, number, would later provide the rational legitimacy for any operations conducted within this concept. Tracing number back to "the mental act of counting" represented the first step in establishing such legitimacy. Another issue in the classical period concerned digits; it was essential to regard digits as pure forms representing numbers to eliminate a digit-based mysticism from the outset and to establish a "relational mathematics." Thus, as a universal science, arithmetic (*arithmetica universalis*) is entirely based on the concept of number; it investigates number-related concepts and the relationships between them. Such an approach eliminates any uncertainties and thus mysticism that may occur within arithmetic. However, while this process imparted a calculative character to Ottoman mathematics, it also hampered studies of number theory. This quest and its historical development, which can be traced back to al-Khwārizmī, will undoubtedly be the subject of our later research.

100 For the origin and process of the problems related to this topic in the context of their relationship with the concepts of continuity and infinity, see Bertrand Russell, *The Principles of Mathematics* (Cambridge: Cambridge University Press, 1903, repr., New York: W.W. Norton, 1996), 325–68.

101 Eves, *Foundations*, p. 178.

102 Carl B. Boyer, *A History of Mathematics* (New York: John Wiley & Sons, 1968), 553–74. For the development of this process and its influence on Husserl, see Miller, *Numbers*, 1–4.

103 Carl B. Boyer, *The History of Calculus and Its Conceptual Development* (Mineola: Dover Publications, 1959), 285.

APPENDIX 1: Unity and Number

[ʿAlī al-Qūshjī, *Sharḥ Tajrīd al-ʿaqāʾid*, ed. Muḥammad Ḥusayn al-Zirāʿī al-Riḍāʾī (Qum, 1393), 1: 514–17]

(والوحدة ليست بعدد) لأنّ العدد – لكونه كمّا – يقبل القسمة، والوحدة لا تقبله. ومن جعلها عدداً أراد بالعدد ما يدخل تحت العدّ، فالنزاع اللفظي.

(بل هي مبدء للعدد المتقوّم بها لا غير) يعني. أنّ كلّ عدد متقوّم بوحداته لا بما دونه من الأعداد؛ فإنّ الستّة مثلاً متقوّمة بالوحدة ستّ مرّات، لا بثلاثة ثلاثة؛ فإنّ تقوّمها بهما ليس بأولى من تقوّمها بأربعة واثنين، ولا من تقوّمها لخمسة واحد. فإن تقوّمت ببعضها لزم الترجّح بلا مرجّح؛ وإن تقوّمت بالكلّ لزم اسغناء الشيء عمّا هو ذاتي له؛ لأنّ كلّ واحد منها كافٍ في تقويمها، فيستغني به عمّا عداه.

فإن قيل: جاز أن يكون كلّ واحد منها مقوّماً لها باعتبار القدر المشترك بين جميعها؛ إذ لا مدخل في تقويمها لخصوصياتها.

قلنا: القدر المشترك بينها الذي يقوّم حقيقة الستّة هو الوحدات، فما ذكر اعتراف المطلوب.

لا يقال: تقوّمها بالوحدات أيضاً ليس بأولى من تقوّمها بالأعداد، فيعود المخذور، أعني: الترجّح بلا مرجّح.

لأنّا نقول: التقوّم بالوحدات راجح باعتبار أنّه لازم على كلّ حال. وأيضاً: يمكن تصوّر كنه كلّ عدد مع الغفلة عمّا دونه من الأعداد؛ فإنّ العشرة مثلاً إذا تُصُوِّرت وحداتها من غير شعور بخصوصيات الأعداد المندرجة تحتها فقد تُصُوِّرت حقيقة العشرة بلا شبهة، فلا يكون شيء من تلك الأعداد داخلاً في حقيقتاها.

(وإذا أضيف إليها مثلها حصلت الاثنينية، وهي نوع من العدد، ثمّ تحصل أنواع لا تتناهى بتزايد واحد واحد)؛ فإنّ الاثنين إذا أضيف إليه وحدة أخرى يحصل ثلاثة، وهي نوع آخر من العدد، وإذا أضيف إليها وحدة أخرى تحصل أربعة، وهي أيضا نوع آخر من العدد، وهكذا كلّ نوع إذا زيد عليه وحدة يحصل نوع آخر، والتزايد لا ينتهي إلى حدّ لا يزاد عليه، فلا ينهتي الأنواع إلى نوع فوقه نوع آخر، **(مختلفة الحقائق هي أنواع العدد)** لاختلاف باللوازم، كالصمّاء

والمنطقية والتركيب والأوّلية؛ واختلاف اللّوازم يدلّ على اختلاف الملزومات.

(وكلّ واحد منها) أي: من أنواع العدد أمر اعتباري، لتقوّمه بالوحدة التي هي **(أمر اعتباري)** لما مرّ من الضابط، **(يحكم به)** أي: بذلك النوع من العدد **(العقل على الحقائق، إذا انضمّ بعضها إلى بعض في العقل انضماماً بحسبه)** أي: بحسب ذلك النوع من العدد، مثلاً إذا انضمّ واحد إلى واحد يحكم العقل بالاثنين عليهما، وإذا انضمّ إليهما واحد آخر يحكم العقل بالثلاثة عليها وهكذا.

(والوحدة قد تعرض لذاتها ومقابلها) فإنّه يقال: وحدة واحدة، و عشرة واحدة؛ فإنّ كلّ ما له وجود – ذهناَ أو خارجاّ – فله وحدة مّا، ولو بالإعتبار، لما سبق من أنّ الوحدة تساوق الوجود.

(ولا يتسلسل) الوحدات **(بل ينقطع بانقطاع الإعتبار)** على ما عرفت أمثالها من الأمور الاعتبارية.

(وقد تعرض لها شركة)؛ فإنّ وحدة زيد تشارك وحدة عمرو في مطلق الوحدة، **(فيتخصّص)** أي: يتميز كلّ منهما عن الأخرى **(بالمشهوري)** أي: بما أضيف هي إليه؛ فإنّ وحدة زيد تمتاز بزيد عن وحدة عمرو، وكذلك وحدة عمرو تمتاز بعمرو عن وحدة زيد، وسيجيء أنّ معروض الإضافة يسمّى مضافاً مشهورياً.

لا يقال: الوحدة نفسها ليست إضافة حتّى يكون معروضا مضافاً مشهورياً، غاية الأمر: أنّه يعرض لها إضافة إلى معروضها.

لأنّا نقول: تلك الإضافة كما تعرض للوحدة وتعرض لموضوعها أيضاً، وبهذا الاعتبار يسمّى موضوعها مضافاً مشهورياً.

وذكروا في شرح هذا المحلّ من المتن ما يقضي من العجب.

(وكذا المقابل) يعني: أنّ الكثرة أيضا تعرض لها شركة، ويتميز عن مشاركها بمعروضها.

(...).

(**Unity is not a number**) because a number—being a quantity—accepts division, whereas unity does not. Those who consider unity to be a number refer to what falls under the act of counting. Therefore, the discussion is merely verbal.

(**Unity is the foundational principle of a number; nothing else.**) That is, every number is composed of unities, not other numbers. For example, six is constructed from "six times unity," not from three plus three. For being constructed from three plus one is no more valid than being constructed from four plus two, or five plus one. If it were constructed from any of them, an arbitrary preference would be required. If it is entirely constructed, then it necessitates the unconditionality of what is essential for it. Each of those is sufficient for the construction of six; thus, there is no need for any others beyond one.

If it is said that each can be a founder based on the common value among all of them, then it need not be that one of them is concretely the founder.

We would say that the common value that establishes the essence of six is the unities, which is the acknowledgment of the result of the syllogism.

It cannot be said: "Being composed of unities is not preferable to being composed of numbers." This implies the return of the issue of "arbitrary choice," which should be avoided.

It can be said: "Being composed of unities" is preferred in every case because it is necessarily required. Similarly, it is possible to conceive the essence of each number without considering other numbers. For instance, the number ten; when one conceives its unities without taking into account the concreteness of the lower numbers, one has certainly conceived its essence, for none of those numbers can be included in its essence.

(**When a single unity is added to another, a pair is formed; this is a type of number. Then, types are obtained where the increment cannot be limited.**) When another unity is added to two (i.e., the pair), three emerges; this is another type of number. When another unity is added, four arises; this is yet another type/kind of number. Thus, when a unity is added to each type, a different type of number is produced. The increase does not end at a limit where it can no longer increase; nor do types find an end at a type that has no higher type above it. Requirements such as irrationality, rationality, compositeness, and primeness vary, as (**the various truths of unity are types of numbers**). The differences in what is required indicate the differences in what is necessary.

(**Each of them**), meaning each of the types of numbers, is conventional because a number is constituted by unity, which is a (**considered concept**), as mentioned previously. (**The mind judges**) the types of numbers (**according to their realities, when one is added to another according to that type in [the] mind**), i.e., according to the kind of that number. For example, when one is added to one, the mind judges that these two are two; when another "one" is added to them, the mind determines that they are three, and this continues accordingly.

(**Unity can also be attributed to its essence and to the multiplicity that opposes it.**) It can be stated: Unity is one; ten is one. Because—whether mental or external—everything that exists, even if conventional, has a type of unity. As previously mentioned, unity accompanies existence (meaning its instances are one).

Unities (**do not form a chain**); as you learned from examples of considered concepts, (**on the contrary, they cease with the cessation of the consideration.**)

Unity can have a commonality; for example, the unity of Z is common with the unity of A in the absolute sense of unity. Thus, **clearly** each is distinguished from the other (**and becomes concrete**); that is, with what is attributed to it... For instance, the unity of Z differs from the unity of A; similarly, the unity of A is differentiated from the unity of Z. As will be mentioned later, the subject of attribution is referred to as "clear attribution."

It cannot be said: "The unity itself is an attribution that has a clear subject of attribution." The intended benefit here is this: that is, in addition to the subject, unity can also be an attribute. It can be said: "The mentioned attribution is an attribute to the unity just as it is to its subject." In this respect, the subject is referred to as clear attribution.

In this part of the commentary, they mentioned things that are surprising.

(**The same applies to the counterpart**) That is, multiplicity is also subject to commonality; however, it is distinguished from its counterparts by its subject.

(...)

APPENDIX 2: Category of Quantity

[ʿAlī al-Qūshjī, *Sharḥ Tajrīd al-ʿaqāʾid*, ed. Muḥammad Ḥusayn al-Zirāʿī al-Riḍāʾī (Qom , 1393), 2: 219]

(**الأوّل: الكمّ**) يريد أن يذكر المباحث المتعلّقة بكلّ واحد من المقولات التسع، فبدأ بالكمّية؛ لأنّها أعمّ وجوداً من الكيفيّة وأصحّ وجوداً من الباقي. أمّا أنّها أعمّ وجوداً من الكيفيّة؛ فلأنّ العدد من الكمّية عارض للأمور المقارنة للكيفيّة – أعنى المادّيات – وعارض أيضاً للمجرّدات العارية عن الكيفيّات؛ فقد وجد الكمّية مع الكيفيّة وبدونها؛ فيكون أعمّ وجوداً منها؛ وكون المجرّدات عالِمَة مثلا لايقتضي كونها معروضة للكيفيّة؛ لجواز أن لا يكون علمها بحصول صور الأشياء فيها. وقد يقال: إنّ العدد يعرض لجميع المقولات حتّى لنفسه، والكيفيّة لا تعرض لنفسها.

وأمّا أنّها أصحّ وجوداً من البافي؛ فلأنّ الباقي أعراض نسبيّة لا تقرّر لها في ذوات موضوعاتها إلّا مقيسة إلى غيرها، بخلاف الكمّية؛ فإنّها متقرّرة في ذوات موضوعاتها مع قطع النظر عمّا عداها.

(**It is the first quantity.**) He began with quantity because he wanted to mention the topics related to the nine categories. The category of quantity is more general than the category of quality (*aʿamm*) and is also more certain (*aṣaḥḥ*) than the other categories. It is more general than the category of quality because a quantity, namely a number, can pertain to both the states adjacent to quality—namely material things—and to abstract entities free from quality. Quantity exists together with quality and things outside of quality, and it is more general than quality in existence. For instance, the knowledge of abstract entities does not require them to possess the qualities of the forms of things, as their existence does not necessitate it. It is said that numbers pertain to all categories; indeed, they pertain even to themselves, while quality cannot pertain to itself. It is more certain than the remaining categories because the other categories are relative accidents that occur only in comparison to one another in their essences. This is contrary to quantity, as quantity is established from the essence of its subjects without considering the other categories outside of itself.

APPENDIX 3: Jamshīd al-Kāshī, *Miftāḥ al-ḥussāb*

[ed. Aḥmad Saʿīd al-Demirdaş and Muḥammad Ḥamdī al-Ḥifnī al-Shaykh (Cairo, n.d.), 44; ed. Nādir al-Nābulusī (Dimashq, 1977), 47]

الحساب عِلْمٌ لقوانين استخراج مجهولاتٍ عدديةٍ من معلوماتٍ مخصوصة.

فموضوعه: العدد، وهو ما يقع في العدّ؛ ويشتمل على الواحد وعلى ما يتألف منه، فهو باعتبار كمّيّته الذاتيّة – أي بكونه غير مضاف إلى جملة – يُسَمّى صحيحاً كالواحد والاثنين والعشرة والخمسة عشر والمائة.

وباعتبار كمّيّته الاضافيّة – أي بكونه مضافاً إلى جملة – يُسَمّى كسراً، والجملة المسوبة إليها تُسَمّى مخرجاً كالواحد من الاثنين وهو النّصف، وكالثّلاثة من الخمسة وهو ثلاثة أخماس الواحد.

والمراد بالكمّيّة ما يقع في جواب «كمّ؟»؛ أو الكمّ الاصطلاحي لا يصدق على الواحد.

Ḥisāb (accounting): It is a science for the rules of obtaining numerical unknowns from known private/given quantities.

Its subject: It is number. A number is that which falls under the act of counting; it encompasses one and those derived from one.

A number, in one regard, is an intrinsic quantity; that is, it exists without reference to a whole. This type of number is called whole numbers, like one, two, ten, fifteen, and one hundred.

In another respect, it is a relative quantity; that is, it exists with reference to a whole. This type of number is named a fraction; the whole it is related to is called the denominator: ½ and 3/5, which is three-fifths of one.

The purpose of quantity is what is given as the answer to the question “How many?” Terminological (*iṣṭilāḥī*) quantity does not denote the individual.

APPENDIX 4: Fanārīzāde ʿAlāʾ al-Dīn ʿAlī Chalabī, *Sharḥ al-Tajnīs fī ʿilm al-ḥisāb*

[III. Ahmet 3154, ff. 1b-2a; 66a-76a]

1و/ اعلم أنّ الحساب علم يعرف منه استخراج مجهولات عدديّة من معلوماتها؛ وموضوعه العدد لكن لا مطلقاً بل من حيث يستعلم منه كيفيّة ذلك الاستخراج.

وأمّا العدد المطلق فهو موضوع العلم العدد المسمّى بآرتماطيقي الذي يبحث فيه عن الأعراض الذاتية للعدد من حيث هو عدد كالزّوجيّة والفرديّة وزوج الزوج وزوج الفرد وزوج الزوج والفرد؛ وهو من أصول الرّياضي. وأمّا الحساب فهو فروعه؛ ويعلم منه كيفيّة الأعمال من الضرب والقسمة والنسبة والتعضعيف والنصنيف وغير ذلك. فاعلم ولا تشكك فيما قلنا وإن اشتبه على كثير من العلماء.

والعدد عند طائفة كميّة تطلق على الواحد وما يتألف منه؛ ولم يريدوا بالكميّة المصطلحة وإلاّ لم يطلقوا 2ظ/ على الواحد بل ما يقع في جواب كم؟

وعند أخرى كميّة تتألف من الوحدات.

واستحسن المصنف أن يقال ما يكون نصف مجموع حاشيته –أي المتساويتي القرب منه؛ فالواحد ليس بعدد على هذا.

وزعم من لا يتحقق له من الفرقة الثانية أنّ الإثنين أيضا ليس من بعدد. ونحن نقتفي إثر الأولى لما ستعرف انشاء الله.

66و/ **والعدد ما هو** – أي الكمّ المنفصل الذي هو القائم بنفسه، –أي المعتبر من حيث هو هو، فإنّه حينئذ يكون مستغنيا في التعقّل عن يفعل ما عداه حيث لم يعتبر معه إضافة إلى الغير بالجذريّة أو الماليّة أو نحو ذلك حتى يتوقّف تعقّله تعقّل ذلك الغير، فكما أنّه قائم بنفسه لا يحتاج إلى غيره. وإلى هذا المعنى أشار بقوله **ومعناه إذا لم يكن مضافاً إلى مال أو جذر** – أي معنا ما ذكر من التعريف أنّه إنّما يسمى عدداً إذا لم يكن مضافاً إلى مال أو جذر أو غيرهما حتى لو اعتبر معه إضافة إلى أحدهما لم يسم عدداً بل شيئاً أو مالاً / .../

66ظ/ الكمّ المنفصل إن اعتبر من حيث هو هو يسمى عدداً.

وقال بعضهم: العدد ما يتركّب من الواحد – أي يحصل من اجتماع مثله أو أمثاله.

وفي معناه البعض الآخر: العدد هو الكثرة المؤلّفة من الوحدات.

ولمّا كان التعريف الأوّل تعريفاً بالأخفى كما لا يخفى، والثّانى محتاجا في صحّته إلى تأويل؛ كما ذكرنا لم يرتض المصنّف شيئا منهما فذكر ما هو المرضى عنده **وأصحّ العبارات** في تعريفه أن يقال ما كان نصف مجموع حاشيتيه المتقابلتين – أي المتساويتي القرب منه / .../

76و/ وأنت تعلم أنّ التعريف الذي ارتضاه المصنّف كالتّعريف الذي قبله إنّما يصحّ على مذهب الثاني من المذاهب الثلاثة التي أشرنا إليه في صدر الكتاب دون الأوّل والثالث.

ومن تصفّح قواعد الفنّ وأنعم النّظر فيها تبيّن له ابتناء أكثرها على مذهب الأوّل. فإذن التعريف الصحيح ما ينطبق على هذا المذهب كما اخترناه هناك.

1b/ Know that *ḥisāb* (accounting) is a science in which the extraction of numerical unknowns from known quantities is learned. Its subject is number; however, not in an absolute sense, but rather in terms of understanding the condition of this extraction itself.

As for absolute numbers, they are the subject of the science of numbers, also called arithmetic, in which the intrinsic properties of numbers—such as evenness, oddness, even-even, even-odd, and even-odd-odd—are studied. Number theory is a branch of mathematics, while accounting is a subfield. With *ḥisāb*, one learns the qualities of operations like multiplication, division, ratio, doubling, and halving. Know these and do not doubt what we say on this matter, which many scholars confuse!

According to one group, a number is referred to as a quantity composed of one and those derived from one. This group does not mean conceptual quantity; otherwise, they would not call it a number. They would refer to whatever answers the question "How many?" Another group says: **2a/** "A number is a quantity made up of units." The author intended to convey a good point by stating: "A number is half the sum of two edges," that is, those equal to it from both sides... Thus, it is not a number. Some members of the second faction, lacking rigorous knowledge, mistakenly believe that two is also not a number. We are, God willing, following the path of the first group.

[...]

66a/ (**A number is that which [is]**) a number, meaning discrete quantity (*al-kamm al-munfasil*)—which exists in itself—[and] refers to what is considered "number-as-number." Therefore, in contemplation (*taʿaqqul*), it is independent of external entities. Its nature is not attributed to others, like rootness or squareness. Its recognition ceases when it is regarded in relation to something else. Since it exists in itself, it does not require anything else. This meaning is indicated by his statement: "Its meaning is that when not attributed to square or root…" In other words, the meaning derived from this definition is that when something is not attributed to square, root, etc., it is called a number. If it is regarded in relation to any of these, it is not called a number; rather, it is named as a thing or square.

66b/ A discrete quantity is referred to as a number when considered "as it is."

Some have said it is something composed of ones, meaning it arises from a single unit or the sum of units.

Others stated, "A number is a multitude made up of units."

The first definition is not obscure; it is formulated with a more implicit concept, while the second requires interpretation for accuracy. Indeed, the author found both definitions insufficient and proposed his own. In his definition, "the most accurate expression" is that "a number is half the sum of two equal edges"; that is, it refers to "those equal from both sides."

67a/ You also know that the definition deemed sufficient by the author is similar to the previous definition. This definition is correct according to the second of the three views we pointed out at the beginning of the book, but not according to the first and third views.

Those who study the rules of this science and delve deeply into it understand that the majority adopt the first view. For this reason, the correct definition is the one that aligns with this view, and here we have also preferred this perspective.

APPENDIX 5: Kātib ʿAlāʾ al-Dīn Yūsuf, *Murshid al-Muḥāsibīn*

(Berlin 2398, ff. 4a-6b).

According to this estimation, the subject of this science is indeed numbers. However, some philosophers argue that the number one is the first number. Others hold differing views, claiming that one is not a number but rather that two is the first number. This disagreement arises because "number is that which is composed of units" and "a number is half the sum of its edges," implying that it cannot be derived from one alone. According to Pythagoras, one is not considered a number because it cannot be multiplied or divided. Therefore, whenever it is multiplied by its own unit, it does not change its form. However, in the case of other numbers, various forms can arise.

For example, if a number is multiplied or divided by its equal or another number, it will increase or decrease. However, if it is flawed, multiplying will not result in an increase and division will not lead to a decrease. It is not contradictory to say that one is not a number; rather, it suggests that all other numbers have their origins, beginnings, limits, and encompassments. Just as, without comparison, the origin of the universe is said to be outside of the universe itself.

However, it is indeed true that it is also a number. For those who say, "A number is that which enters into counting, including one and what is composed of it," are considered experts.

As for those who mean unity by one, it is not appropriate to argue with them. However, if we speak with their terms, [discussion] is possible. When an object is subject to unity, it is referred to as "one." However, if it is subject to a number, it is referred to as "finite." The true attribute of unity is this: "Unity is the attribute which, when applied to a thing, does not divide it."

Now, if "one" (*wāḥid*) and those consisting of one are absolute and not supplemented by anything else, they are considered "correct." If they are not independent, they are called "fractions" (*kasr*).

APPENDIX 6: Taqī al-Dīn al-Rāsid, *Bughyat al-Tullāb min ʿīlm al-ḥisāb*

(Carullah 1454, f. 1b)

الوحدة: ما به يكون الشيء واحدا، وهي صفته.

والواحد من العدد لوقوعه في مراتبه ولخصوصيّات أخرى غير ما موضع من أصول أقليدس.

وقولهم ليس من العدد لأنّ العدد نصف مجموع حاشيته، ليس بشيء.

والكثرة صفة للعدد.

والكثير المتألف من الواحد، ويسمّى بالكمّ المنفصل.

/... /

فإنّ مراتب العدد اعتباريّة لا نهاية لها وإن تناهى المعدود والعاد.

Unity is that which is one with itself; therefore, unity is the attribute of that thing.

[One] is a number due to its various levels (*marātib*) and other properties not mentioned in Euclid's *Elements*.

The statement, "It is not a number because it is half the sum of its margins," is meaningless.

Multiplicity is an attribute of number.

Many are composed of ones and are called discrete quantities.

The levels (*marātib*) of a number are nominal; although both the counter and the counted are limited, there is no boundary/end.

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