NAZARİYAT

Recovering Two Lost Treatises on Approximating the Sine of 1° from Qūshjī's Commentary on *Zīj-i Sulṭānī* *

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Abstract: The accuracy of medieval approximations of the Sine of 1° reached its peak with the works of the Persian mathematician and astronomer Ghiyāth al-Dīn Jamshīd al-Kāshī (d. 832/1429), and his patron Ulugh Beg (d. 853/1449), the Timurid ruler of Transoxiana and a mathematician and astronomer in his own right. Their works were written during the active phase of the Samarkand observatory, which was founded by Ulugh Beg and whose final product, *Zīj-i Sulṭānī*, was the most accurate *zīj* of the medieval period. Even though neither of their treatises on approximating the Sine of 1° has reached us, Kāshī's and Ulugh Beg's approximation methods were transmitted through the works of their colleagues at the observatory, namely Qāḍī-zāda al-Rūmī's (d. after 844/1441) recension of Kāshī's treatise and ʿAlāʾ al-Dīn ʿAlī al-Qūshjī's (d. 879/1474) commentary on *Zīj-i Sulṭānī*. Unlike Qāḍī-zāda's treatise, Qūshjī's commentary has not received the attention it deserves from historians. Thus, it has not been noticed that what is presented in Qūshjī's commentary under the rubric of "Ulugh Beg's demonstrative method" is, in fact, a synthesis of Kāshī's and Ulugh Beg's approximation methods. The present article aims to fill this gap by offering an edition and English translation of the relevant passages of Qūshjī's commentary, distinguishing the two scholars' methods from each other, and presenting them in modern mathematical notation.

Keywords: Late medieval mathematics, Samarkand observatory, astronomical tables, accuracy, approximation, cubic equation, numerical solution, fixed point iteration

* I would like to thank the anonymous reviewers for their constructive suggestions for improving this article. A special note of gratitude goes to Sajjad Nikfahm-Khubravan for his generous help and support at all stages of the research, Osama Eshera for offering invaluable feedback on various iterations of the article, and Hasan Umut for helping with acquiring some of the manuscripts needed for this research.

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DOI dx.doi.org/10.12658/Nazariyat.11.1.M0248en

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Received: 4 June 2024 Accepted: 7 August 2024

Atıf© Savadi, Fateme. "Recovering Two Lost Treatises on Approximating the Sine of 1° from Qūshjī's Commentary on *Zīj-i Sulṭānī* ", *Nazariyat* 11/1 (April 2025): 165-213.

Introduction

When dealing with irrational numbers, ancient and medieval mathematicians invented new approximation algorithms and modified existing ones, with the aim of improving the accuracy of their results. Ptolemy (d. c. 170 CE), for example, sought a more accurate approximation of the chord of $\frac{1}{2}^{\circ}$ in order to present a more accurate chord table in *Almagest* I.11. He calculated the chords for a circle with radius 60 (i.e., the unit circle in the sexagesimal system), starting from $\frac{1}{2}^{\circ}$ and continuing to 180°, at intervals of $\frac{1}{2}^{\circ}$, accurate to two sexagesimal places. Ptolemy's chord table was significantly more accurate than that of his predecessor, Hipparchus (d. c. 120 BC), who had tabulated the chords at intervals of $7\frac{1}{2}^{\circ}$, accurate to one sexagesimal place.

To fill his chord table, Ptolemy started with the arcs whose chords correspond to the sides of inscribed regular polygons and can be easily calculated. Then he applied various theorems to them and calculated the chords of their supplements, differences, halves, and sums, until he filled the chord table at intervals of $1\frac{1}{2}^{\circ}$. However, Ptolemy's goal was to make the table at $1/2$ intervals, so he still had to determine two more chords between each interval of $1\frac{1}{2}^{\circ}$, using the theorem of the sum of two given chords. For this, he needed to determine the chord of $\frac{1}{2}^{\circ}$, which is one third of $1\frac{1}{2}^{\circ}$. But Ptolemy knew that it is impossible to find "by geometrical methods"[1] the chord of an arc that is one third of another arc whose chord is given. This problem goes back to the famous old problem of trisecting an angle using only a compass and an unmarked ruler. Both problems can be reduced to a cubic equation of the kind that cannot be solved by Euclidean geometry (except for a few arcs or angles). So, Ptolemy decided to calculate the chord of $\frac{1}{2}^{\circ}$ by applying the difference theorem to the chords of 1° and $1\frac{1}{2}^{\circ}$. For this calculation, he needed to approximate the chord of 1°, which is also irrational. So, he established a lemma to approximate it using the chords of $1\frac{1}{2}^{\circ}$ and $\frac{3}{4}^{\circ}$, saying "though [the lemma] cannot, in general, exactly determine the sizes [of the chords], in the case of such very small quantities, it can determine them with a *negligibly small error*."[2]

Ptolemy's chord table was preserved and improved by mathematicians in the premodern Islamic world in their handbooks of astronomical tables. Such handbooks, called *zīj*es, were originally adapted from the computational parts of the

1 G.J. Toomer, *Ptolemy's Almagest* (Princeton: Princeton University Press, 1998), 54.

2 Toomer, *Almagest*, 54 (emphasis added).

Almagest, but also included definitions and calculation methods that originated in ancient Indian astronomy, such as Sine, Cosine, Tangent, and Cotangent.[3] More importantly, *zīj*es contained lengthy and accurate tables of all these newly developed trigonometric functions. Due to their practical value, new *zīj*es were continuously being produced, with the goal of increasing accuracy through new computational methods, new observational data, or a combination of the two. Our focus in this article is on the algorithms used to approximate the Sine of 1°, the base value of the Sine table in the *zīj*es upon which all other trigonometrical tables depend. As a result of its significance, mathematicians worked earnestly to improve its approximations. In this respect, the Sine of 1° in the *zīj*es served the same purpose as the chord of ½° in the *Almagest*.

The most accurate medieval tables of trigonometric values appeared in *Zīj-i Sulṭānī*, which was composed by Ulugh Beg (d. 853/1449), the Timurid ruler of Transoxiana and a mathematician and astronomer in his own right. In building the Samarkand observatory, Ulugh Beg's main goal was to compile the most accurate *zīj* of the time, one free from the deficiencies of all previous *zīj*es, especially *Zīj-i Īlkhānī*, which was prepared in 672/1274 at the Maragha observatory. One of Ulugh Beg's collaborators at the observatory, Ghiyāth al-Dīn Jamshīd al-Kāshī (d. 832/1429), had already authored *Zīj-i Khāqānī dar takmīl-i Zīj-i Īlkhānī* in 816/1413 in Kashan, in which he tried to improve upon *Zīj-i Īlkhānī*. One such improvement was the accuracy of the Sine table. Kāshī calculated the Sine table at intervals of one minute for arcs 0–90°, accurate to three sexagesimal places. His Sine table is more accurate than that of *Zīj-i Īlkhānī*, which was calculated for 0–89° at intervals of one minute, and for 90–179° at intervals of one degree, accurate to two sexagesimal places. Still, as we shall see, Kāshī's Sine table was less accurate than what was later achieved in *Zīj-i Sulṭānī*, whose Sine table was calculated for arcs 0–89° at intervals of one minute, accurate to four sexagesimal places.

The approximation of the Sine of 1° that Kāshī used as the base value for his Sine table, namely 1;2,49,43,43, was more accurate than that in *Zīj-i Īlkhānī*, namely 1;2,49,43. However, in an autograph marginal note in the holograph of his *zīj* (copied in 816/1413 in Kashan), Kāshī says:

3 The names of the medieval forms of the trigonometric functions, which were calculated in a sexagesimal system, are capitalized here to distinguish them from the modern functions.

> The ancients could not find a general method of calculating the Sine of one third of an arc whose Sine is given . We invented a method and presented it in a separate treatise. Using that method, we calculated the Sine of 1°; it was 1;2,49,43,11,14,44,16,19,16.[4]

The treatise mentioned by Kāshī in this note has not reached us. However, the list of Kāshī's works given in the preface of his *Miftāḥ al-ḥisāb* includes the title *al-Watar wa-l-jayb*, literally, "on calculating the Sine or chord of an arc the Sine or chord of whose triple is known."[5] Since, *al-Watar wa-l-jayb* has not reached us either, it is not clear if this is the same treatise that Kāshī alluded to in the marginal note in his *zīj* or not. But if it is, then Kāshī must have added the note sometime after he completed *al-Watar wa-l-jayb*, whose composition obviously predated that of the *Miftāḥ* in 830/1427. Moreover, there is evidence to suggest that Kāshī wrote *al-Watar wa-l-jayb* after having completed a work on approximating π, *al-Muḥītiyya*, in 827/1424.

The significance of the *Muḥītiyya* derives not only from Kāshī's astoundingly accurate approximation of π, but also from his thorough discussion of accuracy and error in general.[6] In the preface of the *Muḥītiyya*, Kāshī criticized Abū al-Wafāʾ

4 See Istanbul, Süleymaniye Kütüphanesi, Ayasofya, MS 2692, f. 28b. The marginal note reads:

«و متقدمان طریقه نیافته‌اند کی مطلقاً جیب ثلث قوس معلوم الجیب استخراج توان کردن. ما طریقه استنباط کردیم و در شرح آن رساله علی‌حده نوشتیم. جیب یک درجه بان طریق بیرون آوردیم، بود: ا ب مط مج یا ید مد یو یط یو تاسعه».

As a convention for representing sexagesimal numbers, throughout this article, the whole parts are separated from the fractional parts with a semicolon. For example, 1;2,49,43,11,14,44,16,19,16 is $1 + 2/60 + 49/60^2 + 43/60^3 + 11/60^4 + 14/60^5 + 44/60^6 + 16/60^7 + 19/60^8 + 16/60^9$.

5 It seems that *al-Watar wa-l-jayb* did not enjoy a wide circulation, to the extent that some subsequent scholars speculated that it was never completed. See, for example, the marginal note on the first folio of the earliest copy of the *Miftāḥ*, namely Tehran, Kitābkhāna wa Mūza-yi Millī-yi Malik, MS 3180, which was duplicated in the margin of folio 32a of an early copy of *Zīj-i Khāqānī*, namely London, British Library, India Office, MS Islamic 430. For more information about the codex Tehran, Malik, MS 3180, which is a collection of Kāshī's works, see Osama Eshera, "On the Early Collections of the Works of Ġiyāṯ al-Dīn Jamšīd al-Kāšī," *Journal of Islamic Manuscripts* 13 (2022): 225–62.

6 Kāshī's goal in the *Muḥītiyya* was to minimize the error in calculating the circumference of a circle with a diameter 600,000 times that of the Earth until it is "as negligible as a hair's breadth." To achieve this goal using Archimedes' method, Kāshī showed that the difference between the perimeters of the inscribed and the circumscribed polygons should be less than $1/60^8$ and that such precision will be achieved after 28 bisections of the sides of the inscribed and circumscribed regular triangles, which yield two polygons of 805306368 ($=2^{28}$) sides. By calculating the difference between the perimeters of two such polygons, Kāshī arrived at an approximation of π accurate to 16 decimal places. See Ghiyāth al-Dīn Jamshīd al-Kāshī, *al-Muḥītiyya*, Mashhad, Kitābkhāna-yi Markazī-yi Āstān-i Quds, MS 5389, ff. 1b–2b, 24a. It was not for another two centuries that two sig-

al-Būzjānī (d. c. 998) and Abū Rayḥān al-Bīrūnī (d. c. 1050) for their inaccurate approximations of π. In its final chapter, Kāshī showed how Būzjānī and Bīrūnī could have improved their approximations of π by first improving their approximations of two other irrational values, namely, the chords of ½° and 2°. These two chords correspond to the lengths of the sides of the inscribed polygons of, respectively, 720 sides and 180 sides, used by Būzjānī and Bīrūnī to approximate π according to Archimedes' method.[7]

In order to prove his point, Kāshī offered his method of approximating the chords of ½° and 2°, and came up with two relatively accurate values. However, the approximation of the chord of 2° in the *Muḥīṭiyya*, 2;5,39,26,22,29,28,32,50, is slightly less accurate than the value that can be inferred by doubling the Sine of 1° that Kāshī gave in the marginal note in *Zīj-i Khāqānī*, 2;5,39,26,22,29,28,32,38,32.[8] This means that when writing the *Muḥīṭiyya* in 827/1424, Kāshī had not yet come up with his more accurate value of the Sine of 1°.

Although Kāshī's work is lost, his method of approximating the Sine of 1° has been preserved in the works of his colleagues, the coauthors of *Zīj-i Sulṭānī*, namely Qāḍī-zāda al-Rūmī's (d. after 844/1441) Arabic recension of Kāshī's treatise, written in 836/1432 in Samarkand,[9] and 'Alā' al-Dīn 'Alī al-Qūshjī's (d. 879/1474) Persian

nificantly more accurate approximations were published. They were achieved, using two polygons of 2^{62} sides, by the renowned German mathematician Ludolph Van Ceulen (d. 1610) and were published posthumously: one accurate to 33 decimal places in 1615 and another accurate to 35 places in 1621. During his lifetime, in 1596, Van Ceulen had published a less accurate approximation, to 20 places.

7 See Kāshī, *al-Muḥīṭiyya*, Mashhad, Kitābkhāna-yi Markazī-yi Āstān-i Quds, MS 5389, ff. 1b–2b, 27a–29a.

8 Given that the Sine of an arc is half the chord of twice the arc, from an accurate approximation of the chord of 2° one can easily calculate an accurate approximation of the Sine of 1° and vice versa.

9 For a list of the manuscripts of Qāḍī-zāda'a recension, see Fateme Savadi, *Risāla fī istikhrāj jayb daraja wāḥida, tȧlīf-i Musā ibn Muḥammad Qaḍīzāda Rūmī* (*c. 766–c. 840 AH*) (Tehran: Mīrāth-i Maktūb, 2009), 15–18. Qāḍī-zāda's recension was published as a lithograph in Tehran in 1881–82. A facsimile edition of this lithograph with an English translation and commentary was published in B.A. Rosenfeld and Jan P. Hogendijk, "A mathematical treatise written in the Samarqand observatory of Ulugh Beg," *Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften* 15 (2003): 25–65. I have published a critical edition of Qāḍī-zāda's recension with a modern Persian translation in Savadi, *Risāla fī istikhrāj jayb*, 51–64 and 37–50. For a biography of Qāḍī-zāda, see F. Jamil Ragep, "Qāḍīzāde al-Rūmī: Ṣalāḥ al-Dīn Mūsā ibn Muḥammad ibn Maḥmūd al-Rūmī," in *The Biographical Encyclopedia of Astronomers*, ed. Thomas Hockey et al. (New York: Springer, 2014), 1780–1.

commentary on *Zīj-i Sulṭānī.*[10] Since, Ulugh Beg claimed in *Zīj-i Sulṭānī* that he had discovered "a demonstrative method" for calculating the Sine of 1° and that he had written a treatise about it,[11] the next generation of commentators on his *zīj*, Mīram Čelebī (d. 931/1525)[12] and ʿAbd al-ʿAlī al-Bīrjandī (d. 934/1527–28),[13] followed Qūshjī's path in discussing Ulugh Beg's method. It seems, however, that neither commentator had direct access to Ulugh Beg's treatise, which is now lost, as they relied on Qūshjī's commentary when commenting on Ulugh Beg's claim.[14] Mīram Čelebī and Bīrjandī also discussed Kāshī's method, albeit drawing on Qāḍī-zāda's recension, and thus their commentaries received quite a bit of attention. Mīram Čelebī's commentary, in particular, was subject to several studies and was partially translated to French, German, and Russian.[15]

10 For a detailed biography of Qūshjī, see Hasan Umut, "Theoretical Astronomy in the Early Modern Ottoman Empire: ʿAlī al-Qūshjī's *al-Risāla al-Fatḥiyya*" (PhD diss., McGill University, 2020) 53–112.

11 For an edition and English translation of this passage of *Zīj-i Sulṭānī*, see Appendix 3

12 Mīram Čelebī was the grandson of both Qāḍī-zāda and Qūshjī. For a biography, see İhsan Fazlıoğlu, "Mīram Čelebī: Maḥmūd ibn Quṭb al-Dīn Muḥammad ibn Muḥammad ibn Mūsā Qāḍīzāde," in Hockey et al. *Biographical Encyclopedia*, 1496–8.

13 For a biography, see Takanori Kusuba, "Birjandi: ʿAbd al-ʿAlī ibn Muḥammad ibn Ḥusayn al-Birjandi," Hockey et al., *Biographical Encyclopedia*, 225–6.

14 Rosenfeld and Ahmedov have misattributed Qāḍī-zāda's recension to Ulugh Beg. See A. Ahmedov and B.A. Rosenfeld, "Kto byl avtorom «Traktata ob opredelenii sinusa odnigi gradusa»?" [Who was the author of the "Treatise on the Determination of the Sine of a Degree"?], *Obshchestvennyye nauki v Uzbekistane* 10 (1975): 51–3; "The mathematical treatise of Ulugh Beg," in *Science in Islamic Civilization*, ed. E. İhsanoğlu and F. Günergun (Istanbul: IRCICA, 2000), 143–50. Rosenfeld had once been in favor of Qāḍī-zāda's authorship of the recension, following the position of T.N. Kary-Niyazov and G.D. Dzhalalov, who based themselves on the work of the renowned Ottoman scholar Salih Zeki. See B.A. Rosenfeld and A.P. Yuschkevich, "O traktate Kazi-Zade ar-Rumi ob opredelenii sinusa odnogo gradusa," *Istoriko-Matematicheskie Issledovaniya* 13 (1960): 533–8. In his *Âsâr-ı Bâkiye*, Salih Zeki correctly recognized Qāḍī-zāda as the author of the recension. See Salih Zeki, *Âsâr-ı Bâkiye* (Istanbul: *Matbaa*-i *Âmire*, 1913), 1:133–9. I have shown elsewhere that Qāḍī-zāda's authorship of the recension is indisputable: Fateme Savadi, "Naqdī bar istidlāl-i Ruzinfild dar bāb-i intisāb-i yik risāla-yi riyāḍī bi Ulugh-big" [A criticism of Rosenfeld's argumentation on the attribution of a mathematical treatise to Ulugh Beg], *Tarikh-e Elm: Iranian Journal for the History of Science* 4 (2006): 85–103.

15 For a French translation of Mīram Čelebī's commentary, see L.P.E.A. Sédillot, "De l'algèbre chez les Arabes," *Journal asiatique* 5 (1853): 323–56, reprinted in: Fuat Sezgin, *Islamic Mathematics and Astronomy* (Frankfurt am Main: Institut für Geschichte der Arabisch-Islamischen Wissenschaften, 1998), 55:157–90. Several studies have been undertaken based on Sédillot's translation: F. Woepcke, "Discussion de deux méthodes arabes pour déterminer une valeur approchée de sin 1°," *Journal de Mathématiques Pures et Appliquées* 19 (1854): 153–76, 301–3, reprinted in Sezgin, *Islamic Mathematics and Astronomy* 55:191–217); Hermann Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter* (Leipzig: Teubner, 1874), 289–93; Asger Aaboe, "Al-Kashi's iteration method for the

Qūshjī's commentary, which does not mention Kāshī's name at all, has received far less attention, and thus is the focus of the present article.[16] We show that what is presented in Qūshjī's commentary under the rubric of "Ulugh Beg's demonstrative method," is, in fact, a synthesis of Kāshī's four methods for establishing two equivalent algebraic equations, Ulugh Beg's own two methods for establishing the same equations, and Ulugh Beg's fixed point iteration method for solving one of the equations.[17]

In Qūshjī's commentary, each of the six methods is further divided into steps. The corresponding steps of the different methods are presented in parallel, but not necessarily in sequence. This peculiar arrangement makes it tricky for the reader to follow each method from beginning to end. Thus, in Appendices 4 and 5 of this article, a Persian edition and English translation of the relevant passages from Qūshjī's commentary are presented with some annotations. In the remainder of the article, we shall argue that, in this part of his commentary, Qūshjī drew extensively on Ulugh Beg's now-lost treatise. Then, with the help of Qāḍī-zāda's recension, we will show that four

determination of sin 1," *Scripta Mathematica* 20 (1954): 24–9, reprinted in Sezgin, *Islamic Mathematics and Astronomy*, 56:354–60). German translation: Carl Schoy, "Beiträge zur arabischen Trigonometrie," *Isis* 5 (1923): 364–99, reprinted in Sezgin, *Islamic Mathematics and Astronomy*, 25:150–85. A Russian translation by B.A. Rosenfeld has been published in: Dzhemshid Giyaseddin al-Kashi, *Klyuch Arifmetiki, Traktat ob Okruzhnosti* [*The Key to Arithmetic* and the *Treatise on the Circumference*], ed. V.S. Segal and A.P. Yushkevich, trans. B.A. Rosenfeld, comm. A.P. Yushkevich and B.A. Rosenfeld (Moscow: Gosudarstvennoe izdatel'stvo tekhniko-teoreticheskoi literatury, 1956), 311–19. For a modern study in Ottoman Turkish, see Zeki, *Âsâr-ı Bâkiye*, 1:121–32. For a thorough online literature review, see Jan P. Hogendijk, "The Samarkand school in astronomy and mathematics (ca. 1420 CE)," last modified February 2022, https://www.jphogendijk.nl/samarkand.html. The parts of Bīrjandī's commentary that deal with Kāshī's method have been published by the renowned Iranian historian of science, Abolghassem Ghorbani, in *Kāshānī-nāma: aḥvāl va āthār-i Ghiyāth al-Dīn Jamshīd Kāshānī*, 2nd ed. (Tehran: Markaz-i nashr-i dānishgāhī, 1368 SH [1989–90]), 160–7.

16 One study on the approximation of the Sine of 1° that briefly dealt with Qūshjī's commentary is Fateme Savadi, "Risāla-ī fārsī darbāra-yi muḥāsiba-yi jayb-i yik daraja" [A Persian treatise on the determination of the Sine of 1°], *Tarikh-e Elm: Iranian Journal for the History of Science* 6 (2009): 69–104. The article includes an edition of and commentary on an anonymous Persian treatise on the approximation of the Sine of 1°. The work survives in a unique manuscript (Berlin, Staatsbibliothek, Landsberg, MS 144, ff. 74b-82a) and was based on Qūshjī's commentary and Qāḍī-zāda's recension.

17 It is worth noting that Kāshī's innovative and more accurate iteration method for solving one of the equations has been preserved, with a few mistakes, in Qāḍī-zāda's recension. See Savadi, *Risāla fī istikhrāj jayb*, 30–2 (modern analysis of the method), 47–50 (Persian translation), 61–3 (Arabic text); Rosenfeld and Hogendijk, "A mathematical treatise," 46–9 (English translation and annotations).

of the six methods in Qūshjī's commentary originated in Kāshī's lost treatise, and that Qūshjī quoted them from Ulugh Beg's treatise perhaps without realizing they were Kāshī's. But, before delving into those details, we briefly survey the historical context of Qūshjī's commentary and its composition and transmission history.

Historical Context

In the preface of *Zīj-i Sulṭānī*, Ulugh Beg said that after Kāshī's death at the early stages of the *Zīj*'s composition, followed by Qāḍī-zāda's death, the task of completing the work fell to him and Qūshjī. He described Qūshjī as "the dear son… who, despite his young age and being at the beginning of adulthood, has been the champion in the contest of the arts and sciences, such that undoubtedly the fame of his achievements will, in short order, reach every corner of the world."[18]

In his bio-biographical work *al-Shaqāʾiq al-nuʿmāniyya*, the Ottoman historian Taşköprizâde (d. 968/1561)[19] informs us that Qūshjī was away from Samarkand on a yearslong journey to acquire knowledge and that, upon his return to Samarkand, he dedicated his *Risāla fī ḥall ishkāl muʿaddil al-masīr li-ʿuṭārid* to Ulugh Beg.[20] Evidence for Taşköprizâde's report can be found in Qūshjī's preface to the treatise, wherein he said that he enjoyed Ulugh Beg's company from a young age and was privileged to study philosophical and mathematical sciences with him. While this privilege granted Qūshjī "innumerable benefits," it also made him "the subject of envy." Therefore, he had decided to leave Samarkand.

We do not know when this long journey started and ended, and unfortunately the composition date of *Risāla fī ḥall ishkāl muʿaddil al-masīr li-ʿuṭārid* is not known. Moreover, Qūshjī did not mention in the work's preface the names of the places he visited or in which he resided while writing it.[21] Taşköprizâde, however, tells us that Qūshjī left Samarkand, secretly, for Kerman, where he studied with the local schol-

18 See Ulugh Beg, *Zīj-i Sulṭānī*, Istanbul, Topkapı Saray, Revan, MS 1714, f. 2b.

19 Taşköprizâde studied the mathematical sciences with Mīram Čelebī. See Aḥmad ibn Muṣṭafā Ṭāshkubrī-zāda, *al-Shaqāʾiq al-nuʿmāniyya fī ʿulamāʾ al-dawlat al-ʿuthmāniyya* (Beirut: Dār al-kitāb al-ʿarabī, 1975), 327.

20 Taşköprizâde mistakenly mentioned the Moon, instead of Mercury, as the topic of Qūshjī's work. See Ṭāshkubrī-zāda, *al-Shaqāʾiq*, 98.

21 See George Saliba, "Al-Qūshjī's Reform of the Ptolemaic Model for Mercury," *Arabic Science and Philosophy* 3 (1993): 190–1 for the Arabic and 169–70 for the English.

ars and prepared the draft of his famous commentary on Naṣīr al-Dīn al-Ṭūsī's (d. 672/1274) *Tajrīd al-iʿtiqād*.[22] Given that Qūshjī's commentary on the *Tajrīd* was dedicated to the last Timurid sultan, Abū Saʿīd (r. 855–73/1451–69), we can assume that the work was completed sometime after Abū Saʿīd's ascension to the throne in Samarkand in 855/1452, or after his later ascension in Herat in 861/1457. To avoid positing an unreasonably long period between the draft and the final version of Qūshjī's commentary on the *Tajrīd*, it is preferrable to date Qūshjī's return to Samarkand as closely as possible to Ulugh Beg's death in 853/1449. Thus, we can speculate that Qūshjī's yearslong journey took place after his tenure at the observatory had come to an end and before Ulugh Beg's death in 853/1449.

Qūshjī went to Herat sometime after Abū Saʿīd's ascension to the throne in that city in 861/1457. He lived in Herat for a decade, more or less, before eventually moving to Anatolia.[23] His acquaintance ʿAbd al-Raḥmān al-Jāmī (d. 898/1492), the famous Persian poet and Sufi, helped him in acquiring travel documents by writing letters to Niẓām al-Dīn ʿAlī Shīr Navāʾī (d. 906/1501), an important cultural and political figure in Herat. Jāmī requested that ʿAlī Shīr, in his capacity as keeper of the royal seal (*muhrdār*), facilitate issuing the required documents for Qūshjī and his family, and provide them with security along their way.[24] ʿAlī Shīr had been appointed to this position in 873/1469, when the Timurid prince Ḥusayn Bāyqarā (r. 873–911/1469–1506) established himself as the ruler of Khorasan.[25] This means that 873/1469 can be taken as the *terminus post quem* for Qūshjī's departure from Herat.

Qūshjī was already in Istanbul by 877/1473, during the second reign of the Ottoman sultan Mehmed II (855–86/1451–81).[26] He accompanied Mehmed II in his campaign against the Aq Qoyunlu ruler Uzun Ḥasan (d. 882/1478). The battle was fought near Tercan (modern-day Türkiye); and Mehmed II gained victory on 16 Rabīʿ

22 Ṭāshkubrī-zāda, *al-Shaqāʾiq*, 98.

23 Ḥakīm al-Dīn Idrīs al-Bidlīsī, *Hasht bihisht*, Istanbul, Nuruosmaniye Kütüphanesi, Nuruosmaniye, MS 3212, ff. 30b–31a.

24 A. Urunbaev, *Nāma-hāyi dastnivīs-i Jāmī* (Kabul: Maṭbaʿa-yi dawlatī, 1364 Sh [1985–86]), 107, 122.

25 Subtelny, Maria E., "ʿAlī Shīr Navāʾī," in *Encyclopaedia of Islam Three Online*, ed. K. Fleet, G. Krämer, D. Matringe, J. Nawas and D. J. Stewart (Leiden: Brill, 2011), https://doi.org/10.1163/1573-3912_ei3_COM_23837.

26 Qūshjī dedicated a work on arithmetic, entitled *al-Risāla al-Muḥammadiyya*, to Meḥmed II in Ramaḍān 877/February 1473. See the holograph copy of the *Muḥammadiyya*: Istanbul, Süleymaniye, Ayasofya, MS 2733, ff. 74b and 222a.

al-Awwal 878/August 10, 1473.[27] Qūshjī, who had completed his *al-Risāla al-Fatḥiyya*, on theoretical astronomy a few days earlier (mid-Rabīʿ al-Awwal 878), wrote a brief celebratory note after the colophon of his finalized copy of the work on the victory day, while he was still on the battlefield.[28]

It was perhaps shortly after this event that Qūshjī started the composition of his commentary on *Zīj-i Sulṭānī* in Istanbul. None of the extant copies of the commentary contains a preface or colophon. Even though the commentary covers the whole text of the *Zīj*, it seems Qūshjī never had the chance to finalize it before his death on 8 Shaʿbān 879/December 17–18, 1474.[29]

Manuscripts of Qūshjī's Commentary

The relevant passages of Qūshjī's commentary on *Zīj-i Sulṭānī* II.2, in which Ulugh Beg refers to his method of calculating the Sine of 1°, are edited in Appendix 4 based on five manuscripts: Istanbul, Süleymaniye Kütüphanesi, Carullah Efendi, MS 1493 (MS J); Istanbul, Millet Kütüphanesi, Feyzullah Efendi, MS 1342 (MS F);[30] Istanbul, Kandilli Rasathanesi Kütüphanesi, MS 262 (MS K); Tehran, Kitābkhāna-yi Millī, MS 20127 (MS L); and Ann Arbor, University of Michigan Library, Islamic Manuscripts, MS 827 (MS M). Of these five witnesses, two are dated: MS K was copied in Istanbul

27 V. Minorsky and C.E. Bosworth, "Uzun Ḥasan," in *Encyclopaedia of Islam New Edition Online* (*EI-2 English*), ed. P. Bearman (Leiden: Brill, 2012), https://doi.org/10.1163/1573-3912_islam_SIM_7788.

28 See the holograph copy of *al-Risāla al-Fatḥiyya*: Istanbul, Süleymaniye, Ayasofya, MS 2733, f. 70a. For a transcription and English translation of the note, see Umut, "Theoretical Astronomy," 172–3. For a critical edition, English translation, and commentary on the *Fatḥiyya*, see Umut, "Theoretical Astronomy," part 2, chapter 5 (critical edition: 195–289; critical apparatus: 290–351), chapter 6 (English translation: 353–455), and chapter 7 (commentary: 457–549).

29 For the death date see, Kātib Çelebī, *Sullam al-wuṣūl ilā ṭabaqāt al-fuḥūl*, ed. Ekmeleddin İhsanoğlu et al, 6 vols. (Istanbul: IRCICA, 2010) 2:393.

30 MS F was given as a gift on Friday 27 Jumādā al-Ākhir 891/June 30, 1486 to the Ottoman scholar Muʾayyadzāda ʿAbd al-Raḥmān Efendi (d. 922/1516), who reported the event in a note below a couplet that had been written by the benefactor on f. 1a:

«تحفه‌ای نیست مرا در خور همت لیکن
زین محقر غرضم تذکره‌ی احبابست
حرّره الحقير قطب الفقير»
«أتحفه كاتب هذا البيت أدام الله فضائله إلى فقير عفو الله الصمد عبد الرحمن علي بن مؤيّد غفر لهم ضحوة نهار الجمعة السابع والعشرين من آخر من الجمادين لسنة إحدى وتسعين وثمانمائة هجرية بدار السلطنة قسطنطنية صينت عن البلية»

at the end of Dhū al-Ḥijja 879/beginning of May 1475, four and a half months after Qūshjī died;[31] MS M was copied six years later, at the beginning of Ṣafar 886/April 1481, in Tabriz. In none of the five witnesses is the scribe's name mentioned.

Although the transmission-history of Qūshjī's commentary requires further research, our preliminary analysis of a broader set of manuscripts indicates that certain individuals in Qūshjī's circle were primarily responsible for the work's early dissemination. One such individual is the anonymous scribe of MSS F and L, who is also the scribe of at least one more copy: Istanbul, Süleymaniye Kütüphanesi, Hasan Hüsnü Paşa, MS 1285 (MS S).[32] MS S seems to have been Qūshjī's working copy, as the margins of some folios contain extensive authorial revisions, some of which subsequently appear in the textbody in MSS J and M and in the margin in MS L. For example, the extensive revision of folio 16a of MS S, most likely in Qūshjī's own hand, was transmitted to the margin of MS L (p. 41) and incorporated in the textbody of MSS J (ff. 27b–28b) and M (pp. 49–51). This revision was not incorporated in MS F (f. 20a) or in the earliest dated copy, MS K (f. 168b). Nevertheless, our stemmatological analysis shows that MSS F, K, and L form a family, descending from MS S. Moreover, the anonymous scribe of MSS F, L, and S most likely worked directly with Qūshjī.[33] Unfortunately, MS S could not be used in the appended edition because the relevant passages are missing from the codex; it is nevertheless an important codex, as it contains glosses written by Taqī al-Dīn al-Rāṣid (d. 993/1585) and Mīram Çelebī.[34]

31 The copy of Qūshjī's commentary included in MS K is bound with a copy of *Zīj-i Sulṭānī* in the hand of Qūshjī's student, ʿUmar ibn ʿUthmān al-Ḥusaynī al-Dimashqī. According to an *ijāza* in Qūshjī's hand, glued to the title page of the codex, Dimashqī finished studying the *zīj* with Qūshjī in Jumādā al-Ūlā 879/September–October 1474. Dimashqī's transcription of the *zīj*, perhaps from Qūshjī's copy (*min nuskhati al-aṣli al-maktūbati bi-l-raṣadi al-jadīdi al-sulṭāniyi*), was completed at the end of Jumādā al-Ākhir 879/November 1474, at least one month after the date of the *ijāza*. This suggests that this copy most likely was not the one that was used for studying the *zīj*. For more information on Dimashqī and a transcription of the *ijāza*, see Taha Yasin Arslan, "A Fifteenth-Century Mamluk Astronomer in the Ottoman Realm: ʿUmar al-Dimashqī and his ʿilm al-mīqāt corpus the Hamidiye 1453," *Nazariyat Journal for the History of Islamic Philosophy and Sciences* 4 (2018): 119–40.

32 There is a note on the title page of MS L written by the previous owner of the codex, the Iranian litterateur and historian Jalāl al-Dīn Humāʾī, indicating that he had seen another manuscript of Qūshjī's commentary in the hand of the same scribe. This manuscript has not been located, but most likely, it is distinct from MSS F and S, meaning that this anonymous scribe might have produced four copies of Qūshjī's commentary, if not more.

33 Humāʾī also believed that the two manuscripts that he saw in the hand of our anonymous scribe were copied from Qūshjī's authorized manuscript (*dastūr*).

34 The codex contains at least one marginal note signed by Taqī al-Dīn al-Rāṣid (f. 60a) and two by Mīram Çelebī (ff. 155b and 158b).

MS J also seems to have been copied, fully or partially, during Qūshjī's lifetime, because a short note on folio 1a indicates that parts of the copy were read back to Qūshjī. Based on stemmatological analysis, we can say that MSS J and M descend from the revised MS S (directly or indirectly). The rather erroneous manuscript Bursa, İnebey Kütüphanesi, Genel, MS 4318 (MS B) also seems to descend from MS S.[35]

Appendix 1 contains a provisional stemma for the five manuscripts used in the appended edition. The edition relies mostly on MS L, which is a fine copy made by our anonymous scribe from his earlier MS S, before Qūshjī revised it. MS L was collated with a copy of the revised version and corrected accordingly by someone other than the scribe.

Recovering Kāshī's and Ulugh Beg's Methods from Qūshjī's Commentary

The six methods that are included in Qūshjī's commentary arrive at two equivalent equations, which are exactly the same ones discussed in Qāḍī-zāda's recension of Kāshī's treatise. The algebraic demonstration establishing the equations is based on a geometrical demonstration, which includes two propositions, one from the *Almagest* and the other from the *Elements*. This geometrical demonstration in Qāḍī-zāda's recension comes first, as it should, intuitively and mathematically. In Qūshjī's commentary, however, the algebraic demonstration precedes the geometrical demonstration on which it is based. In fact, this is not the only rearrangement in Qūshjī's commentary that strikes one as mathematically counterintuitive.

There are three direct quotations of Kāshī given in Qāḍī-zāda's recension whose Persian versions appear in dispersed passages of Qūshjī's commentary, correspond-

35 This copy once belonged to the Ottoman scholar ʿAlī al-Riyāḍī, who was active during the reign of Süleyman I (r. 926–74/1520–66). For more information about Riyāḍī, see Fateme Savadi, "Marginalia as a Venue of Debate between Quṭb al-Dīn al-Shīrāzī and His Student Suhrāb ibn Amīr al-Ḥājj," in *Quṭb al-Dīn Shīrāzī in Islamic Intellectual Contexts: Perspectives from the History of Science,* ed. Hasan Umut and Mustakım Arıcı (Istanbul: Ibn Haldun University Press, forthcoming). Riyāḍī's ownership note on f. 280b reads as follows, and is dated 22 Jumādā al-Ūlā 949/September 2–3, 1542:

«استملكه أخفّ عباد الله الراضي علي النسّابة الحقير الرياضي جعل الله مستقبله أحسن حالًا من الماضي في كب د ظمط [= ٢٢ ربيع الثاني ٩٤٩]»

Riyāḍī left a few marginal glosses here and there. An anonymous note on f. 1a mistakenly attributes the transcription of the copy to Qūshjī himself. There is an ownership note by Leys-zāde Pīr Ahmed Çelebi (d. 952/1545) on f. 2a.

ing to different steps of three of the six methods.[36] As we will see below, these three methods are identified as Kāshī's methods in the present article. If we rearrange the passages containing the quotations in Qūshjī's commentary and also other passages that correspond to the same methods according to the order given by Qāḍī-zāda—which he says is the very same order as in Kāshī's original treatise[37]—what emerges is three stylistically coherent and mathematically complete methods. However, it also becomes clear that Qāḍī-zāda's recension does not provide a sufficiently complete presentation of Kāshī's methods, as it only contains a not-so-thorough exposition of them as one method and as it omits parts of Kāshī's geometrical demonstration and misrepresents it. Moreover, there is another piece of evidence that betrays the synthesizer's direct reliance on Kāshī. One of the passages in Qūshjī's commentary preserves Kāshī's value for Sine of 3°, namely 3;8,24,33,59,34,28,14,29, which is the same value quoted in Qāḍī-zāda's recension,[38] while all other passages of Qūshjī's commentary use another value, namely 3;8,24,33,59,34,28,15. This second value, despite lacking the eighth sexagesimal place, is rounded from 3;8,24,33,59,34,28,14,50, which is more accurate than Kāshī's value.

These seemingly disperse cases of explicit overlap between the texts of Qāḍī-zāda's recension and Qūshjī's commentary are significant, especially considering that Qāḍī-zāda attributed the methods to Kāshī while Qūshjī attributed them to Ulugh Beg. More importantly, Qāḍī-zāda made no mention of Ulugh Beg, and Qūshjī none of Kāshī. This means that it was probably Qūshjī's source that first synthetized these methods, and that the source tried to disguise its reliance on Kāshī's treatise. Moreover, It should not be taken for granted that this synthesis was originally done by Qūshjī; rather, we must consider the possibility that he inherited it from his source(s), the most likely candidate being Ulugh Beg's now-lost treatise. Whoever it was who originally put together this synthesis must have had access to Kāshī's lost treatise and consulted it directly, rather than relying on Qāḍī-zāda's recension.

36 Direct quotations from Kāshī are underlined in paragraphs [5], [7], and [23] of Appendices 4 and 5. For these quotations in Qāḍī-zāda's recension, respectively, see Savadi, *Risāla fī istikhrāj jayb*, 47, 45–6, 47 (Persian translation), 61, 59, 60 (Arabic text); Rosenfeld and Hogendijk, "A mathematical treatise," 46, 44, 46 (English translation).

37 See Savadi, *Risāla fī istikhrāj jayb*, 38 (Persian translation), 52 (Arabic text); Rosenfeld and Hogendijk, "A mathematical treatise," 33 (English translation).

38 This value is mentioned in the quotation in passage [7] and can be found in Savadi, *Risāla fī istikhrāj jayb*, 59 (46 of the Persian translation) and in Rosenfeld and Hogendijk, "A mathematical treatise," 44.

One could possibly argue that Kāshī and Ulugh Beg might have arrived at the same methods independently. But this is unlikely on historical grounds, given that Qāḍī-zāda wrote his recension in 836/1432 while the earliest reference to Ulugh Beg's treatise comes in his own *Zīj-i Sulṭānī*, whose holograph dates to 843/1439–40.[39] It is more likely that Ulugh Beg's treatise incorporated material from Kāshī's work, which was written in 827–830/1424–1427.[40] Thus, it is not unreasonable to posit Ulugh Beg as the one who first produced the synthesis that is now preserved in Qūshjī's commentary, especially since it is the only source containing Ulugh Beg's numerical method of solving the equation and his approximation. Moreover, Ulugh Beg had a good reason to blot out any clues to Kāshī's lead in the synthesis. After all, Ulugh Beg claimed that he was the first one to calculate the Sine of 1° "using a demonstrative method."[41]

In the absence of Kāshī's and Ulugh Beg's treatises, all of this speculation remains tentative. But it is not farfetched to suppose, at least, that Qūshjī had a copy of Ulugh Beg's treatise and that he drew on it extensively for this part of his commentary. Obviously, if Qūshjī were aware that four of the methods were originally by Kāshī, he probably would not have attributed all of them to Ulugh Beg. In fact, later commentators, including Qūshjī's and Qāḍī-zāda's grandson, Mīram Čelebī, distinguished between Kāshī's and Ulugh Beg's methods. However, in Qūshjī's commentary, the peculiar arrangement of the synthesis of Kāshī's and Ulugh Beg's methods makes it difficult to distinguish them from each other.

Table 1 shows the extent to which the order of the synthesis differs from that of Qāḍī-zāda's recension (in the case of Kāshī's four methods) and from the mathematically logical structure that one would normally expect. It should be noted that the numbering of the six methods is according to the synthesizer and is preserved in Table 1. Thus, the steps of each method are numbered according to this model:

39 This is the transcription date of the holograph; see Hyderabad, Salar Jung, MS 41, f. 202b. It should be noted, however, that in the preface, in this and in other manuscripts, Ulugh Beg alluded to Qāḍī-zāda's death, which we know occurred after 844/1441. Hence, we can speculate that Ulugh Beg added the preface to this manuscript sometime after the *zīj* was initially completed in 843/1439–40. For a thorough discussion of Qāḍī-zāda's death date based on newly found evidence, see Sajjad Nikfahm-Khubravan, "Ulugh Beg's Treatise on the Altitude Circle" (under review).

40 In an early copy of *Zīj-i Khāqānī*, there is a marginal note written by someone whose name is not fully legible, reporting that after Ulugh Beg's death, he found Kāshī's treatise in Ulugh Beg's book chest. See London, British Library, India Office, MS Islamic 430, f. 32a.

41 For an edition and English translation of the passage of *Zīj-i Sulṭānī* II.2 in which Ulugh Beg refers to his method of calculating the Sine of 1°, see Appendix 3.

KM-2.1, for example, is the first step of Kāshī's second method. In all methods, step zero represents the geometrical demonstration establishing the algebraic equations; the other steps are for simplifying and reducing the algebraic equation to its final form. Qūshjī's commentary contains only Ulugh Beg's iteration method for solving the equation and, in fact, is the only known record of it. The earliest record of Kāshī's numerical solution is Qāḍī-zāda's recension; later sources took it from Qāḍī-zāda.

Steps of methods of Kāshī (KM) and Ulugh Beg (UBM) in logical order	**Corresponding passages in Qūshjī's commentary**[42]
Geometrical preliminaries	[9–10]
KM-2.0 (x=Crd 2°)	[12], [14]
KM-2.1	[18]
KM-2.2	[6]
KM-4.0 (x=Crd 2°)	[23]*
KM-4.1	[8]
KM-1.1 (x=Sin 1°)	[11], [13]
KM-1.1	[17]
KM-1.2	[5]*
KM-3.0 (x=Sin 1°)	[19–22]*
KM-3.1	[7]*
UBM-2.0 (x=Crd 2°)	[16]
UBM-2.1	[4]
UBM-1.0 (x=Sin 1°)	[15]
UBM-1.1	[3]
UBM-1.2, iteration method for solving the equation	[25–26]

Table 1. Logical order of Kāshī's and Ulugh Beg's methods in contrast with corresponding passages in Qūshjī's commentary.

One reason behind the peculiar arrangement of the synthesis in Qūshjī's commentary is that the synthesizer wanted to present the corresponding steps of each method in parallel. This desire will become clear as we go through the modern mathematical representation of each method, starting with our reproduction of the ge-

42 Paragraph [24] of Qūshjī's commentary, which comes just before Ulugh Beg's second method, is not reflected in Table 1 because it merely provides some algebraic terms and definitions. Paragraphs marked with an asterisk contain Persian versions of Qāḍī-zāda's direct quotation from Kāshī.

ometrical diagram of Qūshjī's commentary (Fig. 1). There is a key difference between this diagram and that of Qāḍī-zāda's recension. The small circle and the quadrilateral inscribed in it, which are fundamental for establishing the equation with the Sine of 1° as the unknown, are missing from Qāḍī-zāda's diagram. For this reason, we argue that Qāḍī-zāda misunderstood and criticized Kāshī's geometrical demonstration establishing that equation.[43]

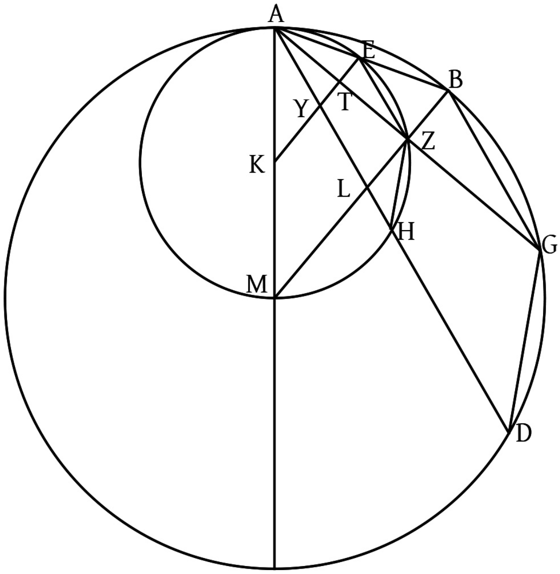


Figure 1. Reproduction of the geometrical diagram of Qūshjī's commentary.

Paragraph [9] sets forth the propositions derived from the *Almagest* and the *Elements*. According to the first proposition, for a quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides. The second proposition states that for two intersecting chords of a circle, the products of the lengths of the line segments on each chord are equal. Paragraph [10] presents the following details about the diagram (Fig. 1):

In the circle *ABGD*: $R = 60,\ \widehat{AB} = \widehat{BG} = \widehat{GD} = 2°$

In the circle *AEZH*: $r = 30,\ \widehat{AE} = \widehat{EZ} = \widehat{ZH} = 1°$

$$\angle BAG = \angle GAD \Longrightarrow \begin{cases} ET = TY \\ BZ = ZL \end{cases}$$

$AZ \perp EK, AZ \perp BM$

43 See Savadi, *Risāla fī istikhrāj jayb*, 46–7 (Persian translation) 59–60 (Arabic text); Rosenfeld and Hogendijk, "A mathematical treatise," 44–6 (English translation).

Using the aforementioned geometrical diagram and propositions, paragraphs [11–12] constitute the first steps in establishing the two main equations: one with the Sine of 1° as the unknown (hereafter, equation 1) and the other with the chord of 2° (hereafter, equation 2). As shown in the table below, in this step, one side of each equation is established:

[11] $AE = EZ = ZH = Sin\ 1° = x$

$AH = Sin\ 3° = 3;8,24,33,59,34,28,15$

In the quadrilateral *AEZH*:

$$AE \times ZH + AH \times EZ = AZ^2 \Longrightarrow$$
$$x^2 + x\ Sin\ 3° = AZ^2 \Longrightarrow$$
$$\boxed{x^2 + 3;8,24,33,59,34,28,15\ x = AZ^2}$$

[12] $AB = BG = BD = Crd\ 2° = x$

$AD = Crd\ 6° = 6;16,49,7,59,8,56,30$

In the quadrilateral *ABGD*:

$$AB \times GD + BG \times AD = AG^2 \Longrightarrow$$
$$x^2 + x\ Crd\ 6° = AG^2 \Longrightarrow$$
$$\boxed{x^2 + 6;16,49,7,59,8,56,30\ x = AG^2}$$

The goal of the second step is to establish the other side of each equation. Paragraphs [13–14] correspond to this step. Then, the third step is to establish the two equations in their complete form. Paragraphs [17] and [18], respectively corresponding to Kāshī's methods 1 and 2, represent how he arrived at equations 1 and 2 from the geometrical preliminaries. Paragraphs [5] and [6] are algebraic re-renderings of Kāshī's methods 1 and 2.

[13] According to the proposition from the *Elements*:

$$AT \times TZ = ET \times (2r - ET)$$
$$\Longrightarrow AT^2 = 2r \times ET - ET^2$$

According to the Pythagorean theorem, in the right triangle *ATE*:

$$AE^2 = AT^2 + ET^2$$
$$and\ AT^2 = 2r \times ET - ET^2$$
$$\Longrightarrow \boxed{x^2 = 2r \times ET}$$

[14] Similarly:

$$AB^2 = AZ^2 + BZ^2$$
$$and\ AZ^2 = 2R \times BZ - BZ^2$$
$$\Longrightarrow \boxed{x^2 = 2R \times BZ}$$

[17] Kāshī's method 1:

$$AT^2 = \frac{1}{4} AZ^2$$

and from [11]:

$$AZ^2 = x^2 + 3;8,24,33,59,34,28,15\ x \Longrightarrow$$
$$AT^2 = \frac{1}{4} x^2 + 0;47,6,8,29,53,37,3,45\ x$$
$$and\ ET^2 = x^2 - AT^2 \Longrightarrow$$
$$\boxed{ET^2 = \frac{3}{4} x^2 - 0;47,6,8,29,53,37,3,45\ x}$$
$$\xrightarrow{\times 4} \boxed{3x^2 - 3;8,24,33,59,34,28,15\ x = EY^2}$$

[18] Kāshī's method 2:

$$AZ^2 = \frac{1}{4} AG^2$$

and from [12]:

$$AG^2 = x^2 + 6;16,49,7,59,8,56,30\ x \Longrightarrow$$
$$AZ^2 = \frac{1}{4} x^2 + 1;34,12,16,59,47,14,7\ x$$
$$and\ BZ^2 = x^2 - AZ^2 \Longrightarrow$$
$$\boxed{BZ^2 = \frac{3}{4} x^2 - 1;34,12,16,59,47,14,7\ x}$$
$$\xrightarrow{\times 4} \boxed{3x^2 - 6;16,49,7,59,8,56,30\ x = BL^2}$$

From [13]:

$$x^2 = 2r \times ET, and\ EY = 2ET \Longrightarrow$$
$$x^2 = r \times EY = 30\ EY \Longrightarrow$$
$$x^4 = 15,0;0\ EY^2 \Longrightarrow$$
$$45,0;0\ x^2 - 47,6;8,29,53,37,3,45\ x = x^4$$
$$etc.$$

[5] Kāshī's method 1, in purely algebraic terms:

$$x^2 - \left(\frac{1}{4}x^2 + 0;47,6,8,29,53,37,3,45\ x\right) = \frac{3}{4}x^2 - 0;47,6,8,29,53,37,3,45\ x$$
$$\xrightarrow{\times 4} 3x^2 - 3;8,24,33,59,34,28,15\ x$$
$$\xrightarrow{\times 15,0;0} 45,0;0\ x^2 - 47,6;8,29,53,37,3,45\ x$$
$$45,0;0x^2 - 47,6;8,29,53,37,3,45\ x = x^4 \Longrightarrow$$
$$45,0;0\ x^2 = x^4 + 47,6;8,29,53,37,3,45\ x$$
$$\xrightarrow{\div 45,0;0}$$
$$x^2 = 0;0,1,20\ x^4 + 1;2,48,11,19,51,29,25\ x$$
$$\xrightarrow{\div x}$$
$$x = 0;0,1,20\ x^3 + 1;2,48,11,19,51,29,25$$

From [14]:

$$x^2 = 2R \times BZ, and\ BL = 2BZ \Longrightarrow$$
$$x^2 = R \times BL = 60\ BL \Longrightarrow$$
$$x^4 = 1,0,0;0\ BL^2 \Longrightarrow$$
$$3,0,0;0\ x^2 - 6,16,49;7,59,8,56,28\ x = x^4$$
$$etc.$$

[6] Kāshī's method 2, in purely algebraic terms:

$$x^2 - \left(\frac{1}{4}x^2 + 1;34,12,16,59,47,14,7\ x\right) = \frac{3}{4}x^2 - 1;34,12,16,59,47,14,7\ x$$
$$\xrightarrow{\times 4} 3x^2 - 6;16,49,7,59,8,56,28\ x$$
$$\xrightarrow{\times 60^2} 3,0,0;0x^2 - 6,16,49;7,59,8,56,28\ x$$
$$3,0,0;0x^2 - 6,16,49;7,59,8,56,28\ x = x^4 \Longrightarrow$$
$$3,0,0;0\ x^2 = x^4 + 6,16,49;7,59,8,56,28\ x$$
$$\xrightarrow{\div 3}$$
$$1,0,0;0x^2 = \frac{1}{3}x^4 + 2,5,36;22,39,42,58,50\ x$$
$$\xrightarrow{\div 60^2}$$
$$x^2 = \frac{1}{3}0;0,1x^4 + 2;5,36,22,39,42,58,50\ x$$
$$\xrightarrow{\div x}$$
$$x = 0;0,0,20\ x^3 + 2;5,36,22,39,42,58,50$$

Paragraphs [7] and [8] represent Kāshī's third and fourth methods in purely algebraic terms. The geometrical demonstration of these two methods is provided in paragraphs [19–23]. Occurrences of Kāshī's value for the Sine of 3°, namely 3;8,24,33,59,34,28,14,29, in paragraph [7] are colored in green.

[19–22]	[7] Kāshī's method 3:	[8] Kāshī's method 4:	[23]
Finding *EY=BZ*	$\text{Sin } 1° = x \xrightarrow{2} x^2$ $\xrightarrow{\div 30} \frac{x^2}{30} = 0;2x^2$ $\xrightarrow{60-(\ )} 60 - 0;2x^2$	$\text{Crd } 2° = x \xrightarrow{2} x^2$ $\xrightarrow{\div 60} \frac{x^2}{60} = 0;1x^2$ $\xrightarrow{120-(\ )} 2;0 - 0;1x^2$	Finding *BL=AK* Finding

<table>
<tr>
<td>Finding $MZ=R-BZ$

Finding $AZ^2=R^2-MZ^2$</td>
<td>$\xrightarrow{60^2-(\ \ )^2}$
$1,0,0;0-(1,0,0;0+0;0,4x^4-4x^2)$
$=4x^2-0;0,4x^4$</td>
<td>$\xrightarrow{120^2-(\ \ )^2}$
$4,0,0;0-(4,0,0;0+0;0,1x^4-4x^2)$
$=4x^2-0;0,1x^4$</td>
<td>$2R-AK$

Finding $AG^2=4R^2-(2R-AK)^2$</td>
</tr>
<tr>
<td colspan="2">[7] Kāshī's method 3, continued:
$4x^2-0;0,4\,x^4=x^2+3;8,24,33,59,34,28,14,29\,x$
$\Longrightarrow$
$3x^2-0;0,4\,x^4=3;8,24,33,59,34,28,14,29\,x$
$\xrightarrow{+0;0,4x^4}$
$3x^2=3;8,24,33,59,34,28,14,29\,x+0;0,4x^4$
$\xrightarrow{\div 3}$
$x^2=0;0,1,20\,x^4+1;2,48,11,19,51,29,25\,x$
$\xrightarrow{\div x}x=0;0,1,20\,x^3$
$+1;2,48,11,19,51,29,25$</td>
<td colspan="2">[8] Kāshī's method 4, continued:
$4x^2-0;0,1x^4=x^2+6;16,49,7,59,8,56,40,30\,x$
$\Longrightarrow$
$3x^2-0;0,1x^4=6;16,49,7,59,8,56,40,30\,x$
$\xrightarrow{+0;0,1x^4}$
$3x^2=0;0,1x^4+6;16,49,7,59,8,56,40,30\,x$
$\xrightarrow{\div 3}$
$x^2=0;0,0,20x^4+2;5,36,22,39,42,58,50\,x$
$\xrightarrow{\div x}x=0;0,0,20x^3+2;5,36,22,39,42,58,50$</td>
</tr>
</table>

Finally, paragraphs [15] and [16] show how Ulugh Beg arrived at his two equations from the geometrical preliminaries, and paragraphs [3] and [4] are their algebraic representations.

[15]

$$ET=\frac{x^2}{2r}=\frac{x^2}{60}$$
$$\Longrightarrow \boxed{ET^2=0;0,1\,x^4}$$
$$and\ AE^2=AT^2+ET^2$$
$$\Longrightarrow x^2=AT^2+0;0,1\,x^4$$
$$AT^2=\frac{1}{4}AZ^2$$

and from [11]:

$$x^2+3;8,24,33,59,34,28,15\,x=AZ^2$$
$$\Longrightarrow$$
$$AT^2=\frac{1}{4}x^2+0;47,6,8,29,53,37,3,45\,x$$
$$and\ ET^2=x^2-AT^2\Longrightarrow$$
$$\boxed{ET^2=\frac{3}{4}x^2-0;47,6,8,29,53,37,3,45\,x}$$

[3]

$$\text{Sin }1^\circ=x\xrightarrow{2}x^2\xrightarrow{\div 60}\frac{x^2}{60}\xrightarrow{2}\frac{x^4}{60^2}=0;0,1\,x^4$$
$$0;0,1\,x^4=\frac{3}{4}x^2-0;47,6,8,29,53,37,3,45\,x$$

[16]

$$BZ=\frac{x^2}{2R}=\frac{x^2}{120}$$
$$\Longrightarrow \boxed{BZ^2=0;0,0,15\,x^4}$$
$$and\ AB^2=AZ^2+BZ^2$$
$$\Longrightarrow x^2=AZ^2+0;0,0,15\,x^4$$
$$AZ^2=\frac{1}{4}AG^2$$

and from [12]:

$$x^2+6;16,49,7,59,8,56,30\,x=AG^2$$
$$\Longrightarrow$$
$$AZ^2=\frac{1}{4}x^2+1;34,12,16,59,47,14,7\,x$$
$$and\ BZ^2=x^2-AZ^2\Longrightarrow$$
$$\boxed{BZ^2=\frac{3}{4}x^2-1;34,12,16,59,47,14,7\,x}$$

[4]

$$\text{Crd }2^\circ=x\xrightarrow{2}x^2\xrightarrow{\div 60}\frac{x^2}{60}\xrightarrow{\div 2}\frac{x^2}{120}\xrightarrow{2}0;0,0,15\,x^4$$
$$0;0,0,15\,x^4=\frac{3}{4}x^2-1;34,12,16,59,47,14,7\,x$$

$\Longrightarrow$	$\Longrightarrow$
$0;0{,}1\,x^4 + 0;47{,}6{,}8{,}29{,}53{,}37{,}3{,}45\,x = \frac{3}{4}x^2$	$0;0{,}0{,}15\,x^4 + 1;34{,}12{,}16{,}59{,}47{,}14{,}7\,x = \frac{3}{4}x^2$
$\xrightarrow{+\left(\frac{LHS}{3}=\frac{RHS}{3}\right)}$	$\xrightarrow{+\left(\frac{LHS}{3}=\frac{RHS}{3}\right)}$
$0;0{,}1{,}20\,x^4 + 1;2{,}48{,}11{,}19{,}51{,}29{,}25\,x = x^2$	$0;0{,}0{,}20x^4 + 2;5{,}36{,}22{,}39{,}42{,}58{,}50\,x = x^2$
$\xrightarrow{\div x} 0;0{,}1{,}20\,x^3 + 1;2{,}48{,}11{,}19{,}51{,}29{,}25 = x$	$\xrightarrow{\div x} 0;0{,}0{,}20\,x^3 + 2;5{,}36{,}22{,}39{,}42{,}58{,}50 = x$

In paragraph [25] Ulugh Beg explained how he was going to solve equation 1 using an iteration method. In paragraph [26], Ulugh Beg started from the initial value 1;2,48,11,19,51,29,25 and, using fixed point iteration, arrived at the final value of 1;2,49,43,11,14,44,16 for the Sine of 1°.

Conclusion

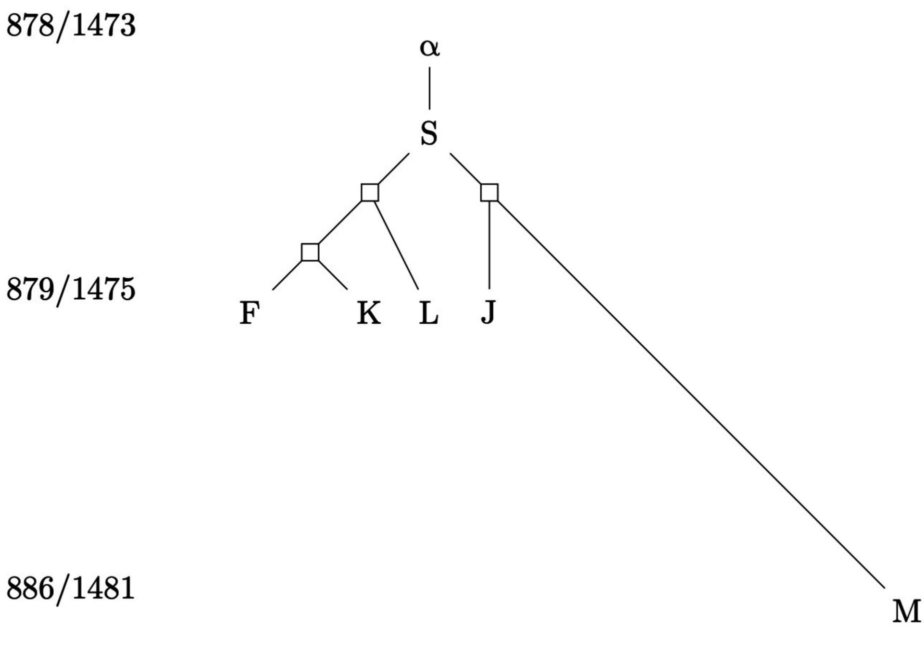
By examining Qūshjī's commentary on Ulugh Beg's statement in *Zīj-i Sulṭānī* about his approximation of the Sine of 1°, and comparing it with Qāḍī-zāda's recension of Kāshī's *al-Watar wa-l-jayb*, we were able to recognize the contribution of each of the four scholars to a collective pursuit of mathematical accuracy. Ulugh Beg founded the Samarkand observatory in an effort to compile the most accurate *zīj* of the time and invited Kāshī to collaborate with him. Kāshī's passion for precision became evident with his composition of the *Muḥiṭiyya* in 827/1424. His efforts toward developing a more accurate approximation of the chords of ½° and 2° in the *Muḥiṭiyya* motivated him to address the broader problem of calculating the Sine or chord of one third of an arc or angle whose Sine or chord is given, in the now-lost *al-Watar wa-l-jayb*, which he wrote sometime between 827/1424 and 830/1427. In 836/1432, four years after Kāshī's death, Qāḍī-zāda wrote a recension of *al-Watar wa-l-jayb*, which he found very concise and perplexing.

Qūshjī started collaborating with Ulugh Beg midway through the *zīj* project until its completion in 843/1439–40. At some point between 836/1432 and 843/1440, Ulugh Beg adapted Kāshī's treatise into a new work, and added to it his modifications to Kāshī's methods. As such, Ulugh Beg's work was a synthesis of his and Kāshī's methods. The political turmoil in Samarkand in the aftermath of Ulugh Beg's death in 853/1449 and the constant rivalry and conflict between Turkoman confederations over Khurasan spurred Qūshjī to migrate at least twice: once from Samarkand to Herat and another time from Herat to Istanbul, where he wrote his commentary on *Zīj-i Sulṭānī* shortly before his death.

In commenting on Ulugh Beg's claim to have discovered a method of accurately approximating the Sine of 1°, Qūshjī drew heavily on Ulugh Beg's now-lost treatise, and this part of his commentary is our only complete record of Ulugh Beg's approximation method. It has also preserved Kāshī's four methods of establishing the equations. Given that Qāḍī-zāda's recension does not provide a sufficiently complete presentation of Kāshī's methods, especially when it comes to the geometrical demonstration, Qūshjī's commentary is a crucial source for recovering Kāshī's methods in their most complete and original form.

Appendix 1

Provisional stemma for Qūshjī's commentary on *Zīj-i Sulṭānī*



Appendix 2

Abbreviations of the critical apparatus

Abbreviation	Description
ها	In the margin (*hāmish*)
شا	Crossed out (*mashṭūb*)
فا	Above the baseline (*fawq al-saṭr*)
تا	Below the baseline (*taḥt al-saṭr*)
[	Separates lemma from variance
:	Separates variance from witnesses in which it appears
=	Separates two variances that share a lemma
-	Omitted
+	Added after the lemma
ج	Istanbul, Süleymaniye Kütüphanesi, Carullah Efendi, MS 1493, ff. 57a, 64b–72b
ف	Istanbul, Millet Kütüphanesi, Feyzullah Efendi, MS 1342, ff. 39b–40a, 45b–51b
ک	Istanbul, Kandilli Rasathanesi Kütüphanesi, MS 262, ff. 178b, 181b–184b
ل	Tehran, Kitābḫāna-yi Millī, MS 20127, pp. 82, 94–106
م	Ann Arbor, University of Michigan Library, Islamic Manuscripts, MS 827 (copied beginning of Ṣafar 886/April 1481 in Tabriz), pp. 97, 110–122

Appendix 3

The lemma from *Zīj-i Sulṭānī* II.2, in which Ulugh Beg refers to his method of calculating the Sine of 1°

و جیب یک درجه که بنای عمل جدول جیب و ظل بر آن است الی یومنا هذا هیچ کس به طریقی[1] برهانی استخراج نکرده و همهٔ حکما تصریح کرده‌اند به آن که[2] طریقی[3] علمی[4] به استخراج آن نیافته‌اند و حیلت کرده‌اند تا به تقریب به دست آورده‌اند. و ما بعنایة الله ومنّه به طریقی برهانی ملهم شدیم و در بیان آن علی‌حده کتابی پرداختیم و به آن جیب برهانی این جداول عمل کردیم.[5]

The Sine of 1°, which is the base [value] for setting out the Sine and Tangent tables, has not been calculated, to this day, using a demonstrative method. All scholars have asserted that they could not find a scientific[ally justifiable] method of calculating it and resorted to tricks for approximating it. We, by virtue of Divine providence and grace, were inspired with a demonstrative method and wrote a separate book on its exposition and set out these [Sine] tables using that [value of the] Sine [of 1°].

Appendix 4

Qūshjī's commentary on the lemma from *Zīj-i Sulṭānī* II.2

[۱] ... حاصل آمد چندین ا ب مط مب یز یه یب، و این جیب یک درجه است به تقریب. این است حیلتی که حکما در استعلام جیب یک درجه کرده‌اند.

1 به طریقی] بطریق: ج.
2 به آن که] با: ج، ک، م = + نکه: فام.
3 طریقی] طریق: ج، ف، ک، ل، م.
4 علمی] عملی: ک.

5 This text is drawn from the manuscripts of Qūshjī's commentary. Cf. L.P.E.A. Sédillot, *Prolégomènes des Tables Astronomiques d'Oloug-Beg* (Paris: Typographie de Firmin Didot frères 1847), 344, reprinted in Sezgin, *Islamic Mathematics and Astronomy*, vol. 52.

[۲] بعد از این طریق برهانی که مصنف قدس سره به آن مهتدی شده بیان کنیم:

[۳] و آن چنان است که جیب یک درجه را شیء فرض کنیم، پس مربع او را که مال است بر شست[6] قسمت کنیم. و مربع خارج قسمت که یک ثانیه مال مال بود مساوی ثلاثه ارباع مال باشد الا این قدر اشیا مز و ح کط نج لز ج مه ثامنه.[7] پس یک ثانیه مال مال و این مقدار اشیا معادل ثلاثه ارباع مال باشد. و چون ثلث هر یک از معادلین را بر وی افزاییم یک ثانیه و بیست ثالثه مال مال و این عدد اشیا، اعنی عدد مذکور مزیداً علیه ثلثه، ا ب مح یا یط نا[8] کط که سابعه معادل باشد با مال. و چون هر یک از معادلین را یک مرتبه حط کنیم، یک ثانیه و بیست ثالثه مکعب و این عدد مذکور معادل باشد با یک شیء.

[۴] و اگر خواهیم وتر دو درجه را شیء فرض کنیم و مربع او را بر شست[9] قسمت کنیم، مربع نصف خارج قسمت که یه ثالثه مال مال است[10] مساوی ثلاثه ارباع مال باشد الا این قدر اشیا ا لد یب یو نط مز[11] ید ز سابعه.[12] پس یه ثالثه مال مال و این مقدار اشیا معادل ثلاثه ارباع مال باشد. و چون ثلث هر یک از معادلین را بر وی افزاییم و یک مرتبه منحط گیریم، ک ثالثه مکعب و این عدد ب ه لو کب لط مب نح ن سابعه معادل یک شیء شود.

[۵] طریقی دیگر: جیب یک درجه را شیء فرض کنیم و ربع مال و این قدر اشیا را مز و ح کط نج لز ج مه ثامنه از مال نقصان کنیم، ثلاثه ارباع مال الا اشیای[13] مذکور[14] باقی ماند. چون این باقی را در چهار ضرب کنیم، سه مال الا

6 شست] شصت: ک، م.
7 ثامنه] بانه: ک.
8 نا] با: ج، م.
9 شست] شصت: ج، ک.
10 است] + ا: ک.
11 مز] مب: ف، ک.
12 سابعه] ثامنه: ل.
13 اشیای] ساء: ج، م.
14 مذکور] مذکوره: ل.

این اشیا ج ح کد لج نط لد کح یه سابعه شود. چون این مبلغ را در یه مرفوع مره ضرب کنند، مه مرفوع مره مال شود الا این اشیا مز و ح کط نج لز ج مه سادسه. و این معادل با مال مال باشد. پس مه مرفوع مره مال معادل بود با یک مال مال و این اشیای مذکور. و چون هر یک از معادلین را بر مه مرفوع مره قسمت کنند، ظاهر شود که یک مال معادل است با یک ثانیه و بیست ثالثه مال مال و این قدر اشیا ا ب مح یا یط نا[15] کط که سابعه.[16] و چون حط هر یک از معادلین کنند، یک شیء معادل شود با یک ثانیه و بیست ثالثه مکعب و عدد مذکور.

[۶] و اگر خواهیم وتر دو درجه را شیء فرض کنیم و ربع مال و این قدر اشیا را ا لد یب یو نط مز ید ز سابعه از مال نقصان کنیم، سه ربع مال الا اشیای مذکور باقی ماند. و چون این باقی را در چهار ضرب کنند، سه مال شود الا این قدر اشیا و یو مط ز نط ح نو کح[17] سابعه. و چون مربع شست را در این مبلغ ضرب کنند، سه مال مرفوع مرتین الا این قدر اشیا و یو مط ز نط ح نو کح[18] خامسه شود.[19] و این معادل مال مال باشد، پس سه مال مرفوع مرتین معادل مال مال و اشیای مذکور باشد. و[20] لازم آید که[21] یک مال مرفوع مرتین معادل ثلث مال مال و این قدر اشیا ب ه لو کب لط مب نح ن خامسه باشد. و چون هر یک از عدیلین را بر مربع شست قسمت کنیم، یک مال معادل ثلث یک ثانیه مال مال و این قدر اشیا ب ه لو کب لط مب نح ن سابعه شود. و[22] باز حط هر یک از عدیلین کنیم، ظاهر شود که یک شیء معادل بیست ثالثه مکعب و عدد مذکور باشد.

15 نا] نا: م.
16 سابعه] فاج.
17 کح] تال = ل: ج، ف، ک، ل، م.
18 کح] ل: ج، ف، ک، ل، م.
19 شود] -ج.
20 و] -ک.
21 که] -م.
22 و] -ج، -ف، -ک، -م.

[۷] طریق دیگر: جیب یک درجه را شیء فرض کنیم و مربع او را که مال است بر سی[23] قسمت کنیم و خارج قسمت را که دو دقیقه مال است از شست[24] نقصان کنیم،[25] باقی ماند شست عدد الا دو دقیقه مال. پس مربع باقی را که ا ۰ ۰ عدداً و چهار ثانیه مال مال است الا چهار مال از مربع شست که ا ۰ ۰ عدداً است نقصان کنیم، باقی ماند چهار مال الا چهار ثانیه مال مال. و این معادل یک مال و این قدر اشیا ج ح کد لج نط لد کح ید کط ثامنه[26] است. و چون یک[27] مال که مشترک است از عدیلین نقصان کنیم، سه مال الا چهار ثانیه مال مال معادل اشیای مذکور باشد. و چون چهار ثانیه مال مال را بر اشیای مذکور افزاییم، معلوم شود که سه مال معادل اشیای مذکور است با چهار ثانیه[28] مال مال. و لازم آید که یک مال معادل یک ثانیه و بیست ثالثه مال مال و ثلث اشیای مذکور باشد،[29] اعنی ا ب مح یا یط نا[30] کط که سابعه. و چون هر یک از عدیلین را یک مرتبه حط کنیم، لازم آید که یک شیء معادل یک ثانیه و بیست ثالثه مکعب و این عدد مذکور باشد، چنانچه به وجوه سابقه لازم آمده بود.

[۸] و اگر خواهیم وتر دو درجه را شیء فرض کنیم و مربع او را که مال است بر شست[31] قسمت کنیم و خارج قسمت را که یک دقیقه مال است از صد و بیست نقصان کنیم، ب ۰ عدداً[32] الا یک دقیقه مال باقی ماند. پس مربع باقی

23 سی] شی: ف، ک.
24 شست] شصت: ج، ک.
25 کنیم] - ج.
26 ثامنه] ثانیه: ل.
27 یک] - ج.
28 ثانیه] + مذکور: شاف.
29 باشد] هاف«صح».
30 نا] نا: ج، ک، م.
31 شست] شصت: ج، ک.
32 عدداً] عدد: ج، ک = + ا: فاک.

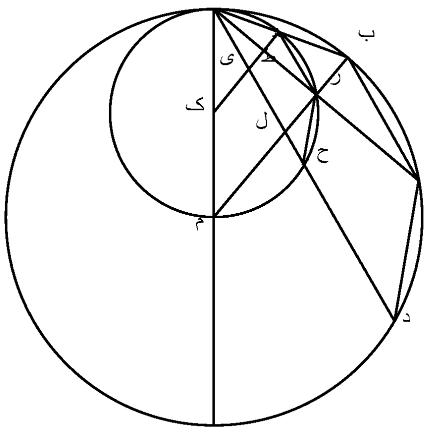
را که د [33] ۰ [34] عدداً و یک ثانیه مال مال است الا چهار مال از د [35] ۰ [36] عدداً نقصان کنیم، باقی ماند چهار مال الا یک ثانیه مال مال، و این مساوی یک مال[37] و این قدر اشیا و یو مط ز نط ح نو ل سابعه باشد. و چون یک مال را که میان عدیلین مشترک است از عدیلین نقصان کنیم و مستثنی[38] را که یک ثانیه مال مال است بر عدیلین افزاییم، ظاهر شود که سه مال معادل یک ثانیه مال مال و اشیای مذکور است. و چون هر یک از معادلین را به ثلث رد کنیم، معلوم شود که یک مال معادل بیست ثالثه مال مال و ثلث اشیای مذکور است، اعنی ب ه لو کب لط مب نح ن سابعه. و چون حط هر یک از معادلین کنیم، ظاهر شود که یک شیء معادل بیست ثالثه مکعب و عدد مذکور است.[39]

[۹] برهان بر این جمله مبتنی بر دو مقدمه است، یکی[40] در مجسطی مبین است و دیگری در اقلیدس. اما مقدمهٔ مجسطی آن است که هر ذو اربعه اضلاع که در دایره‌ای واقع شود، چون متقابلین از این چهار ضلع را مسطح کنند، مجموع این دو مسطح[41] مساوی باشد با مسطح دو قطر این ذی اربعه اضلاع. و مقدمهٔ اقلیدس آن است که هر دو وتر که در دایره‌ای تقاطع کنند، مسطح دو قسم یک وتر مساوی بود با مسطح دو قسم وتر دیگر.[42]

[۱۰] و بعد از تقدیم این دو مقدمه، دایرهٔ **ابجد** بر مرکز **م** رسم کنیم و هر یک از قوس **اب بج جد** به قدر دو درجه فصل کنیم و اوتار **اب بج جد اج**

33 ۰] فال.
34 ۰] -ج، -ف، -ک، -م.
35 ۰] فال.
36 ۰] -ج، -ف، -ک، -م.
37 مال] + را که مان: شاج.
38 مستثنی] سسی: ک.
39 مذکور است] مذکوست: ج.
40 یکی] -م.
41 مسطح] سطح: ف، ک، ل.
42 وتر دیگر] و نزدی کتر: ک.

اد وصل کنیم و قطر **ام** اخراج کنیم. و بر منتصف **ام**، اعنی بر نقطه **ک**، نصف دایرۀ **ام** رسم کنیم، لا محاله اوتار **اب اج اد** را بر نقط[43] **ه ز ح** تنصیف کند به جهت آن که اقطاری که از نقطۀ **م**[44] بر این نقط[45] سه‌گانه آید عمود باشد بر هر یک از این اوتار سه‌گانه به شکل سی‌ام از مقالۀ سیوم.[46] و هر یک از[47] سه قوس **اه ءز زح** دو درجه از دایرۀ خرد باشد زیرا که نسبت اوتار این قسی با نصف قطر دایرۀ خرد چون نسبت اوتار قسی دایرۀ بزرگ است با نصف قطر او. و[48] نصف قطر[49] **بزم** اخراج کنیم تا وتر **اد** را بر **ل** قطع کند. و همچنین نصف قطر **که** اخراج کنیم تا وتر **از** را بر **ط** تنصیف کند زیرا که از مرکز به منتصف قوس آمده و وتر **اح** را بر **ی** قطع کند.[50] و **هط** مساوی **طی** باشد و **بز** مساوی **زل** زیرا که دو زاویۀ **باج** و **جاد** متساویانند به شکل بیست و ششم از مقالۀ سیوم. و خط **از** عمود است بر هر یک از دو خط **هک بم** به شکل سیوم از مقالۀ سیوم.



43 نقط] نمطه: ج، ک.
44 م] + برىن ىمطه م: شاک.
45 نقط] نمطه: ج، ک.
46 سیوم] سیم: ج.
47 از] ازىن سه قوس: ک.
48 با نصف قطر او. و] هام.
49 او و نصف قطر] -ج.
50 کند] کنند: ج.

[۱۱] پس در دایرهٔ خرد[51] به وصل **هز زح** ذی اربعه اضلاع **اهزح** واقع شود. و **اه** جیب یک درجه باشد و **اح** جیب سه درجه باشد. پس به حکم مقدمهٔ مجسطی، سطح **اه** در **زح**، اعنی مربع **اه**، با سطح **اح** در **هز**، مجموع این هر دو مساوی بود با مربع **از**. و چون جیب یک درجه را شیء فرض کنند، در ذی اربعه اضلاع **اهزح**، سطح **اه** در **زح** مال بود و سطح **هز** در **اح** اشیا بود به عدد جیب سه درجه، اعنی ج ح کد لج[52] نط لد کح یه سابعه. و[53] مجموع این هر دو مسطح[54] مساوی بود با مربع قطر **از**.

[۱۲] و به حکم همین مقدمه، چون وتر دو درجه را شیء فرض کنند، در ذی اربعه اضلاع **ابجد**[55] سطح **اب** در **جد** مال بود و سطح **بج** در **اد** اشیا بود[56] به عدد وتر شش درجه، اعنی و یو مط ز نط ح نو ل سابعه. و مجموع این هر دو مسطح[57] مساوی بود با مربع قطر **اج**.

[۱۳] و به حکم مقدمهٔ اقلیدس، مربع **اط** مساوی بود با سطح **هط** در تمام او از قطر دایرهٔ خرد. و مربع **اه** که جیب یک درجه است که مال فرض کرده‌ایم، مساوی بود با سطح **هط** در قطر دایرهٔ خرد، به جهت آن که مربع **اه** به حکم شکل عروس مساوی بود با مجموع مربع **اط** و مربع **هط**.

[۱۴] و به حکم همین مقدمهٔ اقلیدس، مربع **اب** که مال فرض کرده‌ایم مساوی بود با[58] سطح **بز** در قطر دایرهٔ بزرگ.

51 خرد] خردتر: ک.
52 لج] مج: ج.
53 و] -ک.
54 مسطح] سطح: ج، ف، ک، م.
55 اب‌جد] ابدج: م.
56 بود] + بر: شاج.
57 مسطح] سطح: ک.
58 با] فاج.

[۱۵] و چون این مقدمات مقرر شد، آن که گفته است[59] در طریق اول که مربع جیب یک درجه را که مال است بر شست[60] قسمت کنیم و مربع خارج قسمت مساوی ثلاثه ارباع مال باشد الا این قدر اشیا مز و ح کط نج لز ج مه ثامنه،[61] وجهش آن است که مبین شد[62] که مربع **اه** مساوی سطح **هط** در قطر دایرۀ خرد است. و قطر دایرۀ خرد چون مساوی نصف قطر دایرۀ بزرگ است، شست[63] درجه باشد. پس خارج قسمت مال بر شست که یک دقیقه مال باشد مقدار خط **هط** باشد. و مربع خط **هط** که یک ثانیه مال مال باشد با مربع **اط** مساوی مال است به حکم شکل عروس. لیکن مربع **اط** ربع مربع **از** است، پس مساوی بود با ربع مال و ربع عدد اشیایی که مربع **از** مشتمل بر آن است، اعنی[64] مز و ح کط نج لز ج مه ثامنه. و چون از مال نقصان کنند، مربع خط **هط** مساوی ثلاثه ارباع مال باشد الا اشیای مذکور. و باقی اعمال ظاهر است کسی را که بر اعمال جبر و مقابله واقف است.

[۱۶] و اما آن که گفته است[65] وتر دو درجه را شیء فرض کنیم و مربع او را بر شست قسمت کنیم، مربع نصف خارج قسمت که یه[66] ثالثه مال مال است مساوی ثلاثه ارباع مال باشد الا این قدر اشیا ا لد یب یو نط مز ید ز سابعه، وجهش آن است که مربع خط **اب** که[67] مال است مساوی سطح[68] خط **بز** است در قطر دایرۀ بزرگ. پس چون بر شست که نصف قطر دایرۀ بزرگ است قسمت کنند، یک دقیقه مال که ضعف خط **بز** است خارج شود. پس خط

59 است] + که: شال.
60 شست] شصت: ک.
61 ثامنه] تاج.
62 شد] شده: ج، ف، ک، م.
63 شست] شصت: ل.
64 اعنی] + نامه و حون ار مال: شاف.
65 است] + که: شاف.
66 یه] -ج.
67 که] -ل.
68 سطح] مسطح: م.

بز سی ثانیه مال باشد و مربع او یه ثالثه مال مال باشد. و چون با مربع **از** مساوی مال است به حکم شکل عروس، باید که چون مربع **از** را از مال نقصان کنند، آنچه ماند مساوی مربع **بز** باشد. لیکن مربع **از** ربع مربع **اج** است، پس مساوی بود با ربع مال و ربع عدد اشیایی که مربع **اج** مشتمل بر آن است. پس لازم آید که مربع **بز** که یه ثالثه مال مال است مساوی ثلاثه ارباع مال الا اشیای مذکوره باشد. و باقی اعمال به[69] برهان جبر و مقابله ظاهر است.

[۱۷] و اما آن که در طریق دوم گفته است که جیب یک درجه را شیء فرض کنیم و ربع مال و اشیایی که ذکر کرده شده از مال نقصان کنیم، ثلاثه ارباع مال الا اشیای[70] مذکور[71] شود، وجهش آن است که مربع خط **اط** مساوی ربع مربع **از** است، اعنی یک مال و این قدر اشیا ج ح کد لج نط لد کح یه سابعه. پس چون ربع مال و ربع اشیای مذکور، اعنی مز و ح کط نج لز ج مه ثامنه، را از مال[72] نقصان کنند، ثلاثه ارباع مال الا[73] این اشیای مذکور باقی ماند و این مساوی مربع خط **هط** باشد. و چون این باقی را در چهار ضرب کنند، سه[74] مال الا این اشیا ج ح کد لج نط لد[75] کح یه سابعه حاصل شود. و این مربع خط **هی** باشد. و مقرر کرده‌ایم که سطح خط **هط** در قطر دایرهٔ خرد که شست است مساوی مال است. پس سطح **هی** که ضعف **هط** است در سی درجه که نصف قطر دایرهٔ خرد است مساوی مال باشد. و چون مربع **هی** را در مربع سی درجه، اعنی یه مرفوع مره، ضرب کنند، مه مرفوع مره مال شود الا این اشیا مز و ح کط نج لز ج مه سادسه. و این مبلغ مساوی مال مال باشد. و باقی اعمال به[76] برهان جبر و مقابله ظاهر است.

69 به] -ک.
70 اشیای] ساء: م.
71 مذکور] مذکوره: ل.
72 مال] -ف، -ک.
73 الا] -ج.
74 سه] -ک.
75 لد] لو: ج.
76 به] -ف، -ک.

[۱۸] و اما آن که گفته است که وتر دو درجه را شيء فرض کنيم و ربع مال و اشيايي که ذکر کرده از مال نقصان کنيم، ثلاثه ارباع مال الا اشياى مذکور[77] شود، وجهش آن است که مربع خط **از** ربع مربع[78] خط **اج** است، اعنى يک[79] مال[80] و اين قدر اشيا و يو مط ز نط ح نو ل سابعه. پس چون ربع مال و ربع اشياى مذکور اعنى ا لد يب يو نط مز يد ز سابعه را از مال نقصان کنند، سه ربع مال الا اشياى مذکور باقي ماند. و اين مساوى مربع خط **بز** باشد. و چون اين[81] باقي را در چهار ضرب کنند، سه مال شود الا اين قدر اشيا و يو مط ز نط ح نو ل. و اين مساوى مربع خط **بل** که ضعف خط **بز** است باشد. و مقرر کرده بوديم که سطح[82] **بز** در قطر دايرۀ بزرگ که صد و بيست است مساوى مال است. پس سطح خط **بل** در نصف قطر که شست است هم مساوى مال[83] باشد. و چون مربع خط **بل** را در مربع شست[84] ضرب کنند، مساوى مال مال باشد. و باقي[85] اعمال ظاهر است.

[۱۹] و اما آن که در طريق سيوم گفته است که مال را بر سى قسمت کنيم، براى آن گفته است که مقدار خط **ەی** را معلوم کند و بيانش در اين زودى گذشت. و آن که گفته است از شست نقصان کنيم، براى آن گفته است که وتر تمام قوس **از** تا نصف معلوم شود، اعنى خط **مز**. زيرا که **ەی** برابر **بز** است به جهت آن که نسبت **اه** با **اب** چون نسبت **ەی** است با **بل**، ليکن **اه** نصف **اب** است، پس **ەی** نصف **بل** اعنى **بز** باشد. و آن که گفته است که مربع باقي را از مربع شست نقصان کنيم، براى آن گفته که مربع خط **از** را معلوم

77 مذکور] مذکوره: ج.
78 خط **از** ربع مربع] هال«صح».
79 يک] + يک: ک = + ز: فاک.
80 مال] -ک.
81 اين] ان: ج.
82 سطح] مسطح: ج.
83 است. پس سطح خط ب ل در نصف قطر که شست است هم مساوى مال] -م.
84 شست] شصت: ج.
85 و باقي] -ک.

کند. چه مربع وتر هر قوسی با مربع وتر تمام آن[86] قوس تا به نصف مساوی مربع قطر است به شکل عروس، و قطر دایرهٔ خرد شست است چنانچه گذشت. این بود بیان آن چه تعلق به هندسه داشت در این وجه.

[۲۰] اما بیان آن چه تعلق به جبر و مقابله دارد: آن[87] که گفته است که مربع شست عدد الا دو دقیقه مال ا ۰ ۰ عدداً و چهار ثانیه مال مال است الا چهار مال، بیانش آن است که در علم جبر و مقابله مبرهن شده که چون جبری[88] را که استثنا در او واقع شده در مثل او ضرب کنند، مضروب مستثنی منه در[89] مستثنی[90] منه[91] و مضروب مستثنی در مستثنی هر دو را جمع کنند و مضروب مستثنی در مستثنی منه و مضروب مستثنی منه در مستثنی این هر دو را از مجموع نقصان[92] کنند، باقی حاصل ضرب باشد. پس به حکم این مقدمه، مضروب شست در شست را که ا ۰ ۰ عدداً[93] است با مضروب دو دقیقه مال در نفس خودش که چهار ثانیه مال مال است جمع کنند. و از مجموع، مضروب مستثنی منه که شست[94] است در مستثنی که دو دقیقه مال است و آن دو مال باشد با مضروب عکس آن که هم دو مال باشد، نقصان کنند. حاصل ضرب آن باشد که گفته است.

[۲۱] و اما آن که گفته که چون ا ۰ ۰ و چهار ثانیه مال مال الا چهار مال را از ا ۰ ۰ عدداً نقصان کنند،[95] چهار مال الا چهار ثانیه مال مال باقی ماند،[96] بنا

86 آن] فاک.
87 آن] + است: ج، ف، ک، ل، م.
88 جبری] عددی: ج، ف، ک، ل، م.
89 در] + مضروب: شاف.
90 مستثنی] + در مسبی: شاف.
91 در مستثنی منه] هال«صح».
92 نقصان] کم: ج، م.
93 عدداً] عدد: ج.
94 شست] شصت: ل.
95 کنند] + باقی: ف، ک، ل.
96 ماند] اند: ج.

بر آن است که در علم جبر و مقابله معلوم شده که چون جبری را که در او استثنا واقع شده از دیگری نقصان کنند، باید که مستثنی را بر منقوص[97] منه افزایند، بعد از آن مستثنی منه را به تمام از او نقصان کنند. پس به حکم این مقدمه، مستثنی را که چهار مال است بر منقوص منه که ا . . عدداً[98] است افزودیم، ا . .[99] که در جانب منقوص منه است به نقصان مثل او تمام ساقط شود و چهار مال باقی ماند. و چون چهار ثانیه مال مال از او نقصان کنند، چهار مال الا چهار ثانیه[100] مال مال باقی ماند چنانچه گفته است.

[۲۲] و اما آن که گفته است که این معادل یک مال و این قدر اشیا ج ح کد لج نط لد کح ید کط[101] است، بیانش آن است که این مساوی مربع خط **از** است. و در این زودی بیان کرده‌ایم که مربع خط **از** مساوی یک مال و اشیای مذکور است. و باقی اعمال ظاهر است.

[۲۳] و اما آن که گفته که[102] مال را بر شست قسمت کنیم، برای آن گفته که مقدار خط **ب ل** معلوم کند. و آن که گفته که خارج قسمت را از صد و بیست نقصان کنیم، برای آن گفته که وتر تمام قوس **اج** را تا به نصف می‌خواهد[103] معلوم کند.[104] و آن که گفته که مربع باقی را از[105] د . .[106] عدداً نقصان کنیم، برای آن گفته که مربع وتر قوس **اج** را معلوم کند. چه در این زودی گذرانیدیم که مجموع مربع وتر قوس با مربع وتر تمام آن قوس تا به نصف مساوی مربع قطر است به شکل عروس. و آن که گفته که این مساوی یک مال و این

97 منقوص] منقصوص: ج.
98 عدداً] عدد: ج.
99 ا . .] -م.
100 ثانیه] سانه: ج.
101 کط] -ج، -م.
102 که] + خارج: شاک = + ز: فاک.
103 خواهد] خوانند: ج.
104 کند] کنند: ج.
105 از] در: ل.
106 .] -ج، -ف، -ک، -م.

قدر اشیا است و یو[107] مط ز نط ح نو ل سابعه، بیان آن در مقدمۀ مجسطی مذکور شده. و باقی اعمال ظاهر است.

[۲۴] و چون قوسی که **اه** جیب او است ثلث قوسی فرض کرده‌ایم[108] که **اح** جیب او است و به براهین مذکور مقرر شد که جیب **اه** معادل یک ثانیه وبیست ثالثه مکعب خودش با ثلث جیب **اح** است، پس ثلاثه امثال جیب **اه** معادل چهار ثانیه مکعب جیب **اه** با جیب **اح** باشد. پس جیب **اح** از ثلاثه امثال جیب **اه** کمتر باشد به چهار ثانیه مکعب جیب **اه**. پس جیب هر قوسی کمتر باشد از ثلاثه امثال جیب ثلث خودش به مضروب مکعب جیب ثلث در چهار ثانیه.

[۲۵] و چون مقرر شد که مجهول که آن وتر دو درجه است یا جیب یک درجه و ما آن را شیء فرض کرده‌ایم، معادل اجزای مکعب آن شیء است با عددی معین، به جهت استعلام این شیء عددی را[109] که با اجزای[110] مکعب معادل شیء است، شیء نامعدل گوییم. و مکعب او را اگر مجهول وتر دو درجه باشد در بیست ثالثه و اگر مجهول جیب یک درجه باشد در یک ثانیه و بیست ثالثه ضرب کنیم. و حاصل ضرب را بر شیء نامعدل افزاییم تا شیء قریب به مقصود حاصل آید. باز مکعب این شیء را در اجزای مذکوره[111] ضرب کنیم و محصول را بر شیء نامعدل افزاییم تا شیء دیگر اقرب حاصل آید. و بر این موجب مره بعد اخری عمل می‌کنیم تا شیء که تالی است مساوی شیء مقدم شود. پس آن شیء محقق و معدل باشد و آن وتر دو درجه یا جیب یک درجه باشد.

[۲۶] و ما به جهت استعلام جیب یک درجه شیء نامعدل را که ا ب مح یا یط نا کط که سابعه است مکعب ساختیم، حاصل شد ا ح مح ل کج لو ل مب یز مج تاسعه. پس این مکعب در ا[112] ک ثالثه ضرب کردیم، حاصل شد

107 یو] نو: ج.
108 کرده‌ایم] کنىم: ک.
109 را] -م.
110 با اجزای] باحراء: ج، ل = + ا: فال.
111 مذکوره] مذکور: م.
112 ا] -م.

. . ا لا مد م لا کح ما[113] ثامنه. پس این حاصل را بر شیء نامعدل افزودیم، حاصل شد شیء قریب[114] ا ب مط مج د لب . نج ما ثامنه. باز این شیء قریب را مکعب ساختیم، حاصل شد ا ح نج لب د ج ن نط[115] ید نز تاسعه. پس این مکعب را در ا[116] ک ثالثه ضرب کردیم، حاصل شد . . ا لا نا کب مه که ح ثامنه. و این حاصل را بر شیء نامعدل افزودیم، حاصل شد شیء اقرب ا ب مط مج یا ید ن ح ثامنه. مکعب این حاصل گرفتیم، شد ا ح نج لب کو ز . کب ی ثامنه. پس[117] این مکعب را در ا ک ثالثه ضرب کردیم، حاصل شد . . ا لا نا کج ید مط ک ثامنه. و حاصل را بر شیء نامعدل افزودیم، حاصل شد شیء اقرب از اقرب ا ب مط مج یا ید مد ید ک ثامنه. باز این حاصل را مکعب ساختیم، حاصل شد ا ح[118] نج لب کو ح لز ه لد ثامنه. پس این مکعب را در ا ک ثالثه ضرب کردیم، حاصل شد . . ا لا نا کج ید[119] نا کط ثامنه. باز این حاصل را بر شیء نامعدل افزودیم، حاصل شد آنچه مقصود است، ا ب مط مج یا ید مد یو سابعه، به جهت آن که بر این قرار یافت. زیرا که مکعب این عدد را چون در ا ک ثالثه ضرب کنی و حاصل را بر شیء نامعدل افزایی، همین عدد بعینه حاصل شود بی هیچ تفاوت. پس این جیب یک درجه است.

113 با تاسعه. پس این مکعب در ا ک ثالثه ضرب کردیم حاصل شد . . ا لا مد م لا کح ما] -ک.
114 قریب] اقرب: م.
115 نط] بط: ک.
116 ا] -م.
117 پس] -م.
118 ح] ج: ج، ف، ک، ل، م.
119 ید] ند: م.

Appendix 5

Translation of Qūshjī's commentary on the lemma from *Zīj-i Sulṭānī* II.2

[1] …it became this amount: 1;2,49,42,17,15,12, which is approximately the Sine of 1°. The preceding was [an account of] the approximative methods that have been used by scholars for finding the Sine of 1°.

[2] Now we shall explain the demonstrative method that the author—may his soul be sanctified—discovered.

[3] The [method] is like this: Let us assume the Sine of 1° to be unknown.[1] Then we divide its square—which is called *māl*—by 60.[2] The square of the quotient, which is 0;0,1 of the square of the square of the unknown,[3] is equal to ¾ of the square of the unknown minus 0;47,6,8,29,53,37,3,45 of the unknown. Thus, 0;0,1 of the square of the square of the unknown plus [0;47,6,8,29,53,37,3,45 of] the unknown is equal to ¾ of the square of the unknown. If we add one third of each side of the equation to itself, 0;0,1,20 of the square of the square of the unknown plus 1;2,48,11,19,51,29,25 unknowns—which is the aforementioned [0;47,6,8,29,53,37,3,45] plus its third—is equal to the square of the unknown. If we lower each side of the equation by one degree, 0;0,1,20 of the cube of the unknown plus the aforementioned [1;2,48,11,19,51,29,25] is equal to the unknown.[4]

[4] If we assume the chord of 2° to be unknown, and divide its square by 60, the square of half the quotient—which is 0;0,0,15 of the square of the square of the unknown—is equal to ¾ of the square of the unknown minus 1;34,12,16,59,47,14,7 unknowns. Thus, 0;0,0,15 of the square of the square of the unknown plus this amount of unknowns is equal to ¾ of the square of the unknown. If we add one third of each side of the equation to itself, then lower it by one degree, 0;0,0,20 of the cube of the unknown plus this number, 2;5,36,22,39,42,58,50, is equal to the unknown.[5]

1 The original word used for the "unknown" is *shayʾ*, which literally means "thing." In the context of the mathematical tradition of the premodern Islamic world, *shayʾ*, in general, refers to the "unknown quantitative thing."

2 The technical word for the square of the unknown in the mathematical tradition of the premodern Islamic world is *māl*.

3 The technical word for the square of the square of the unknown in the mathematical tradition of the premodern Islamic world is *māl-i māl* (Persian) or *māl al-māl* (Arabic).

4 Ulugh Beg: establishing the equation with the Sine of 1° as the unknown.

5 Ulugh Beg: establishing the equation with the chord of 2° as the unknown.

[5] Another method is to assume the Sine of 1° to be unknown. [If] we subtract ¼ of the square of the unknown plus 0;47,6,8,29,53,37,3,45 of the unknown from the square of the unknown, the remainder is ¾ of the square of the unknown minus the aforementioned unknowns. If we multiply this remainder by 4, it becomes 3 squares of the unknown minus 3;8,24,33,59,34,28,15 unknowns. If this result is multiplied by 15,0;0, then it becomes 45,0;0 squares of the unknown minus 47,6;8,29,53,37,3,45 unknowns. This is equal to the square of the square of the unknown. Thus, 45,0;0 squares of the unknown is equal to the square of the square of the unknown plus the aforementioned [47,6;8,29,53,37,3,45] unknowns. If each side of the equation is divided by 45,0;0, it is clear that the square of the unknown is equal to 0;0,1,20 of the square of the square of the unknown plus 1;2,48,11,19,51,29,25 unknowns. If each side of the equation is lowered by one degree, one unknown is equal to 0;0,1,20 of the cube of the unknown plus the aforementioned [1;2,48,11,19,51,29,25].[6]

[6] If we assume the chord of 2° to be unknown, and subtract ¼ of the square of the unknown plus 1;34,12,16,59,47,14,7 unknowns from the square of the unknown, the remainder is ¾ of the square of the unknown minus the aforementioned unknowns. If this remainder is multiplied by 4, it becomes three squares of the unknown minus 6;16,49,7,59,8,56,28 unknowns. If this result is multiplied by the square of 60, it becomes 3,0,0;0 squares of the unknown minus 6,16,49;7,59,8,56,28 unknowns. This is equal to the square of the square of the unknown. Thus, 3,0,0;0 squares of the unknown are equal to the square of the square of the unknown plus the aforementioned [6;16,49,7,59,8,56,28] unknowns. [From this] it follows that 1,0,0;0 square of the unknown is equal to ⅓ of the square of the square of the unknown plus 2,5,36;22,39,42,58,50 unknowns. If we divide each side of the equation by the square of 60, the square of the unknown is equal to ⅓ of 0;0,1 of the square of the square of the unknown plus 2;5,36,22,39,42,58,50 unknowns. Then, once again, [if] we lower each side of the equation by one degree, it is clear that the unknown is equal to 0;0,0,20 of the cube of the unknown plus the aforementioned [2;5,36,22,39,42,58,50].[7]

[7] Another method: Let us assume the Sine of 1° to be unknown and divide its square—which is called *māl*—by 30, and subtract the quotient—which is 0;2 of the square of the unknown—from 60. The remainder is 60 minus 0;2 of the square of the unknown. We then subtract the square of the remainder—which is the number 1,0,0;0 plus 0;0,4 of the square of the square of the unknown minus 4 squares of the

6 Kāshī's method 1: establishing the equation with the Sine of 1° as the unknown.

7 Kāshī's method 2: establishing the equation with the chord of 2° as the unknown.

unknown—from the square of 60—which is the number 1,0,0;0. The remainder is 4 squares of the unknown minus 0;0,4 of the square of the square of the unknown. This is equal to the square of the unknown plus 3;8,24,33,59,34,28,14,29 unknowns. If we eliminate the square of the unknown from both sides of the equation—which they have in common—3 squares of the unknown minus 0;0,4 of the square of the square of the unknown is equal to the aforementioned [3;8,24,33,59,34,28,14,29] unknowns. If we add 0;0,4 of the square of the square of the unknown to the aforementioned [3;8,24,33,59,34,28,14,29] unknowns, it becomes clear that 3 squares of the unknown are equal to the aforementioned [3;8,24,33,59,34,28,14,29] unknowns plus 0;0,4 of the square of the square of the unknown. It follows [from this] that the square of the unknown is equal to 0;0,1,20 of the square of the square of the unknown plus ⅓ of the aforementioned [3;8,24,33,59,34,28,14,29] unknowns—i.e., 1;2,48,11,19,51,29,25. If we lower each side of the equation by one degree, it follows that one unknown is equal to 0;0,1,20 of the cube of the unknown plus this aforementioned [1;2,48,11,19,51,29,25], as entailed from the previous methods.[8]

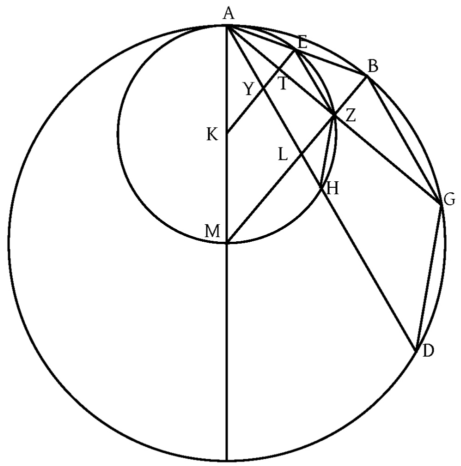
[8] If we assume the chord of 2° to be unknown and divide its square—which is called *māl*—by 60, and subtract the quotient—which is 0;1 of the square of the unknown—from 120, the remainder is the number 2;0 minus 0;1 of the square of the unknown. We then subtract the square of the remainder—which is the number 4,0,0;0 plus 0;0,1 of the square of the square of the unknown minus 4 squares of the unknown—from the number 4,0,0;0. The remainder is 4 squares of the unknown minus 0;0,1 of the square of the square of the unknown. This is equal to the square of the unknown plus 6;16,49,7,59,8,56,40,30 unknowns. If we eliminate the square of the unknown—which both sides of the equation have in common—from both sides of the equation, and add the subtracted term—which is 0;0,1 of the square of the square of the unknown to both sides of the equation, it becomes clear that 3 squares of the unknown is equal to 0;0,1 of the square of the square of the unknown plus the aforementioned [6;16,49,7,59,8,56,40,30] unknowns. If we divide both sides of the equation by 3, it is clear that the square of the unknown is equal to 0;0,0,20 of the square of the square of the unknown plus ⅓ of the aforementioned [6;16,49,7,59,8,56,40,30] unknowns—i.e., 2;5,36,22,39,42,58,50. If we lower each side of the equation by one degree, it becomes clear that the unknown is equal to 0;0,0,20 of the cube of the unknown plus this aforementioned [2;5,36,22,39,42,58,50].[9]

8 Kāshī's method 3: establishing the equation with the Sine of 1° as the unknown.

9 Kāshī's method 4: establishing the equation with the chord of 2° as the unknown.

[9] The demonstration of all of this is based on two propositions: one proved in the *Almagest*, the other in [the *Elements* of] Euclid. The proposition from the *Almagest* is that in any quadrilateral [inscribed] in a circle, when each two opposite sides of the four are multiplied, the sum of the two products is equal to the product of the two diameters of the quadrilateral. The proposition from [the *Elements* of] Euclid is that for any two chords intersecting in a circle, the product of two segments of one chord is equal to the product of two segments of the other chord.

[10] After presenting these two propositions, let us draw the circle *ABGD* with *M* as the center. We then mark the arcs *AB*, *BG*, and *GD*, each measuring 2°, and connect the chords *AB*, *BG*, *GD*, *AG*, and *AD*. We also draw the diameter [passing through the points] *A* [and] *M* and draw the semicircle *AM* around the point *K*, the midpoint of the line *AM*. [The semicircle] certainly bisects the chords *AB*, *AG*, and *AD* at the points *E*, *Z*, and *H*, because the diameters extending from the point *M* to this trinary of points are perpendicular to this trinary of chords, according to the thirtieth proposition of the third book [of the *Elements*]. Each of the arcs *AE*, *EZ*, and *ZH* are two degrees of the small circle [i.e., the semicircle], because the ratio of the chords of these arcs to the radius of the small circle is as the ratio of the chords of the great circle's arcs to its radius. Let us draw the radius *BZM* meeting the chord *AD* at *L*. We also draw the radius *KE* bisecting the chord *AZ* at *T*—as it is extended from the center to the midpoint of the arc—and meeting the chord *AH* at *Y*. *ET* is equal to *TY*, and so is *BZ* to *ZL*, because the angles *BAG* and *GAD* are equal according to the twenty-sixth proposition of the third book [of the *Elements*]. The line *AZ* is perpendicular to each of the two lines *EK* and *BM*, according to the third proposition of the third book [of the *Elements*].



[11] Thus, in the small circle, by connecting [the points] *E* to *Z* and *Z* to *H*, the quadrilateral *AEZH* occurs. [The chord] *AE* is the Sine of 1°, and [the chord] *AH* the Sine of 3°. Therefore, according to the proposition from the *Almagest*, the sum of the product of *AE* and *ZH*—which is the square of *AE*—and of the product of *AH* and *EZ* is equal to the square of *AZ*. If the Sine of 1° is assumed to be unknown, in the quadrilateral *AEZH*, the product of *AE* and *ZH* is the square of the unknown and the product of *EZ* and *AH* is the unknown times the value of the Sine of 3°, namely 3;8,24,33,59,34,28,15, and the sum of these two products is equal to the square of the diameter [of the quadrilateral *AEZH*,] *AZ*.

[12] According to the same proposition, if the chord of 2° is assumed to be unknown, in the quadrilateral *ABGD*, the product of *AB* and *GD* is the square of the unknown and the product of *BG* and *AD* is the unknown times the value of the chord of 6°, namely 6;16,49,7,59,8,56,30, and the sum of these two products is equal to the square of the diameter [of the quadrilateral *ABGD*,] *AG*.

[13] According to the proposition from the [*Elements* of] Euclid, the square of *AT* is equal to the product of *ET* and its complementary segment on the diameter of the small circle. The square of [the chord] *AE*, which is the Sine of 1°, and which we assume to be the square of the unknown, is equal to the product of *ET* and the diameter of the small circle because, according to the Pythagorean theorem, the square of *AE* is equal to the sum of the square of *AT* and the square of *ET*.

[14] According to the same proposition from the [*Elements* of] Euclid, the square of *AB*, which we assume to be the square of the unknown, is equal to the product of *BZ* and the diameter of the great circle.

[15] Now that the preliminaries are established, [recall that] it has been said in the first method, "we divide the square of the Sine of 1°—which is called *māl*—by 60. The square of the quotient is equal to ¾ of the square of the unknown minus 0;47,6,8,29,53,37,3,45 of the unknown."[10] This is because of what has been demonstrated [earlier,][11] that the square of *AE* is equal to the product of *ET* and the diameter of the small circle; the diameter of the small circle is 60, since it is equal to the radius of the great circle. Thus the quotient of the division of the square of the unknown by 60—which is 0;1 of the square of the unknown—is the measure of the line *ET*; the

10 See the beginning of [3].

11 See [11] and [13].

square of the line *ET*—which is 0;0,1 of the square of the square of the unknown—plus the square of *AT* is equal to the square of the unknown, according to Pythagorean theorem. On the other hand, the square of *AT* is ¼ of the square of *AZ*, and thus, it is equal to ¼ of the square of the unknown plus ¼ of the number of the unknown that is part of the square of *AZ*, namely 0;47,6,8,29,53,37,3,45. If this is subtracted from the square of the unknown, the square of the line *ET* is equal to ¾ of the square of the unknown minus the aforementioned [0;47,6,8,29,53,37,3,45 of the] unknown. The remaining operations are clear for anyone who knows algebraic operations.

[16] Now, it has been said, "let us assume the chord of 2° to be unknown and divide its square by 60, the square of the quotient—which is 0;0,0,15 of the square of the square of the unknown—is equal to ¾ of the square of the unknown minus 1;34,12,16,59,47,14,7 unknowns."[12] This is because the square of the line *AB*, which is the square of the unknown, is equal to the product of the line *BZ* and the diameter of the great circle; thus, when it is divided by 60—which is the radius of the great circle—the result is 0;1 square of the unknown, which is twice the line *BZ*. Therefore, the line *BZ* is 0;0,30 of the square of the unknown and, thus, its square is 0;0,0,15 of the square of the square of the unknown. Since, according to the Pythagorean theorem, [this square] plus the square of [the line] *AZ* is equal to the square of the unknown, when the square of [the line] *AZ* is subtracted from the square of the unknown, the remainder is certainly equal to the square of [the Line] *BZ*. On the other hand, the square of [the line] *AZ* is ¼ of the square of [the line] *AG*, and thus, is equal to ¼ of the square of the unknown plus ¼ of the number of the unknowns that are part of the square of *AG*. From this, then, if follows that the square of *BZ*, which is 0;0,0,15 of the square of the square of the unknown, is equal to ¾ of the square of the unknown minus the aforementioned [1;34,12,16,59,47,14,7] unknowns. The remaining operations are clarified through algebraic demonstration.

[17] Now, it has been said in the second method, "let us assume the Sine of 1° to be unknown and subtract ¼ of the square of the unknown plus the aforementioned [0;47,6,8,29,53,37,3,45 of the] unknown from the square of the unknown, it becomes ¾ of the square of the unknown minus the aforementioned [0;47,6,8,29,53,37,3,45 of] the unknown."[13] This is because the square of the line *AT* is equal to ¼ of the

12 See the beginning of [4].

13 See the beginning of [5].

square of the line *AZ*, which is the square of the unknown plus 3;8,24,33,59,34,28,15 unknowns. Thus, when ¼ of the square of the unknown plus ¼ of the aforementioned [3;8,24,33,59,34,28,15] unknowns—i.e., 0;47,6,8,29,53,37,3,45—is subtracted from the square of the unknown, ¾ of the square of the unknown minus the aforementioned [0;47,6,8,29,53,37,3,45 of the] unknown remains, which is equal to the square of the line *ET*. When this remainder is multiplied by 4, the result is 3 square of the unknown minus 3;8,24,33,59,34,28,15 unknowns, which is the square of the line *EY*. We have determined that the product of the line ET and the diameter of the small circle—which is 60—is equal to the square of the unknown. Thus, the product of *EY*—which is twice *ET*—and 30;0—which is the radius of the small circle—is equal to the square of the unknown. Thus, when the square of EY is multiplied by the square of 30;0—i.e., 15,0;0—the result is 45,0;0 squares of the unknown minus 47,6;8,29,53,37,3,45, which is equal to the square of the square of the unknown. The remaining operations are clarified through algebraic demonstration.

[18] Now, it has been said, "let us assume the chord of 2° to be unknown and subtract ¼ of the square of the unknown plus the aforementioned [1;34,12,16,59,47,14,7] unknowns from the square of the unknown; the remainder is ¾ of the square of the unknown minus the aforementioned [1;34,12,16,59,47,14,7] unknowns."[14] This is because the square of the line AZ is ¼ of the square of the line *AG*, i.e., the square of the unknown plus 6;16,49,7,59,8,56,30 unknowns. Thus, when ¼ of the square of the unknown plus ¼ of the aforementioned [6;16,49,7,59,8,56,30] unknowns—i.e., 1;34,12,16,59,47,14,7—is subtracted from the square of the unknown, ¾ of the square of the unknown minus the aforementioned [1;34,12,16,59,47,14,7] unknowns remains, which is equal to the square of the line *BZ*. If this remainder is multiplied by 4, it becomes 3 squares of the unknown minus 6;16,49,7,59,8,56,30 unknowns, which is equal to the square of the line *BL*, which is twice the line *BZ*. We have determined that the product of [the line] *BZ* and the diameter of the great circle—which is 120—is equal to the square of the unknown. Therefore, the product of the line *BL* and the radius [of the great circle]—which is 60—is also equal to the square of the unknown. Thus, when the square of the line *BL* is multiplied by the square of 60, it is equal to the square of the square of the unknown. The remaining operations are clear.

14 See the beginning of [6].

[19] Now,[15] it has been said in the third method, "let us divide the square of the unknown by 30."[16] This is to determine the measure of the line *EY*, whose explanation has been mentioned earlier.[17] As for what has been said, "subtract [the quotient] from 60;"[18] it is to determine the chord of the complement of the arc *AZ* to the half [of the circle], i.e., the line *MZ*, since *EY* is equal to *BZ*, because the ratio of *AH* to *AB* is like the ratio of *EY* to *BL*. On the other hand, *AH* is half of *AB*; thus, *EY* is half of *BL*, which is [equal to] *BZ*. As for what has been said, "subtract the square of the remainder from the square of 60,"[19] it is to determine the square of the line *AZ*, since the square of any arc's chord plus the square of the chord of that arc's complement to the half is equal to the square of the diameter according to the Pythagorean theorem, and the small circle's diameter is 60, as has been mentioned. This was an exposition of what is related to geometry in this method.

[20] Now, the explanation of what is related to algebra: It has been said, "the square of 60 minus 0;2 of the square of the unknown is the number 1,0,0;0 plus 0;0,4 of the square of the square of the unknown minus 4 squares of the unknown."[20] This is because it has been made clear in the science of algebra that when an algebraic expression that contains subtraction is multiplied by itself, the term from which the subtraction is made multiplied by itself is added to the subtracted term multiplied by itself, minus the sum of the product of the subtracted term and the term from which the subtraction is made and the product of the term from which the subtraction is made and the subtracted term; the remainder is the product. Thus, according to this preliminary, the product of 60 and 60—which is the number 1,0,0;0—plus the product of 0;2 of the square of the unknown multiplied by itself—which is 0;0,4 square of the square of the unknown—are added [to each other]. Then, from this sum, [two products] are subtracted: [1] the product of the term from which the subtraction is made—which is 60—and the subtracted term—which is 0;2 of the square of the unknown—and is equal to 2 squares of the unknown, and [2] their product in reverse order, which is also 2 squares of the unknown. The final product is what has been said.

15 The geometrical demonstration of [7] starts in this paragraph and continues until [22].
16 See [7].
17 See [17].
18 See [7].
19 See [7].
20 See [7].

[21] Now, it has been said that when "1,0,0;0 plus 0;0,4 of the square of the square of the unknown minus 4 squares of the unknown is subtracted from the number 1,0,0;0, the remainder is 4 squares of the unknown minus 0;0,4 of the square of the square of the unknown."[21] This is due to what has been made clear in the science of algebra, that when an algebraic expression that contains subtraction is subtracted from another, the subtracted term should be added to the minuend, but the term from which the subtraction is made, in its entirety, should be subtracted from it. Thus, according to this preliminary, we added the subtracted term—which is 4 squares of the unknown—to the minuend—which is the number 1,0,0;0. The 1,0,0;0 that is on the minuend's side is eliminated by the subtraction of its like, and 4 squares of the unknown remain. When, from this, 0;0,4 of the square of the square of the unknown is subtracted, 4 squares of the unknown minus 0;0,4 of the square of the square of the unknown remains, as has been said.

[22] Now, it has been said, "this is equal to the square of the unknown plus 3;8,24,33,59,34,28,14, 29 unknowns."[22] Its explanation is that this [sum] is equal to the square of the line *AZ*, and we have mentioned earlier that the square of the line *AZ* is equal to the square of the unknown and the aforementioned [3;8,24,33,59,34,28,14, 29] unknowns, and the remaining operations are clear.

[23] Now,[23] it has been said, "we divide the square of the unknown by 60." This is to determine the line *BL*. As for what has been said, "we subtract the quotient from 120," it is to determine the chord of the complement of the arc *AG* to the half [of the circle]. As for what has been said, "we subtract the square of the remainder from the number 4,0,0;0," it is to determine the square of arc *AG*'s chord, since we have mentioned earlier[24] that the sum of the square of [any] arc's chord plus the square of the chord of that arc's complement to the half is equal to the square of the diameter, according to the Pythagorean theorem. And the explanation of what has been said, that "this is equal to the square of the unknown plus 6;16,49,7,59,8,56,40,30 unknowns," was mentioned in the course of explaining the proposition from the *Almagest*. The remaining operations are clear.

21 See [7].

22 See [7].

23 This paragraph is the geometrical demonstration of [8].

24 See [19].

[24] Since we have assumed that the arc whose Sine is *AE* is ⅓ of the arc whose Sine is *AH*, and [since,] on account of the aforementioned demonstrations, it has been established that the Sine of *AE* is equal to 0;0,1,20 of its cube plus ⅓ of the Sine of *AH*, therefore, thrice the Sine of *AE* is equal to 0;0,4 of the cube of the Sine of *AE* plus the Sine of *AH*. Therefore, the Sine of *AH* is less than thrice the Sine of *AE* by the amount of 0;0,4 of the cube of the Sine of *AE*. Thus, the Sine of any arc is less than thrice the Sine of ⅓ of itself by the amount of the cube of the Sine of ⅓ [of itself] multiplied by 0;0,4.

[25] Now, it has been established that what is to be known, which is either the chord of 2° or the Sine of 1° and which we assumed to be an unknown, is equal to a fraction of the cube of the unknown plus a certain number. In order to find the value of this unknown, we call the number that together with a fraction of the cube of the unknown is equal to the unknown, the "unmodified [value of the] unknown." We multiply the cube of [the unmodified value] by 0;0,0,20, if the unknown is the chord of 2°, or by 0;0,1,20, if the unknown is the Sine of 1°. We add the product to the unmodified [value of the] unknown to acquire [a value for the] unknown that is close to the desired value. Again, we multiply the cube of this [unmodified value of the] unknown by the aforementioned fractions [i.e., either 0;0,0,20 or 0;0,1,20] and add the product to the unmodified [value of the] unknown to acquire yet another [value for the] unknown that is closer [to the desired value]. We proceed in this manner iteratively until the subsequent [value of the] unknown is equal to the antecedent [value of the] unknown; and thus, this [value of the] unknown is unquestionable and modified, and it is either the chord of 2° or the Sine of 1°.

[26] In order to find the Sine of 1°, we cubed the unmodified [value of the] unknown, which is 1;2,48,11,19,51,29,25, and the result was 1;8,48,30,23,36,30,42,17,43. Then, we multiplied this cube by 0;0,1,20, and the product was 0;0,1,31,44,40,31,28,41. We then added this product to the unmodified [value of the] unknown, and it became close [to the value of the] unknown, [namely,] 1;2,49,43,4,32,0,53,41. Again, we cubed this close [value to the] unknown, and the result was 1;8,53,32,4,3,50,59,14,57. Then, we multiplied this cube by 0;0,1,20, and the product was 0;0,1,31,51,22,45,25,8. We then added this product to the unmodified [value of the] unknown, and it became closer [to the value of the] unknown, [namely,] 1;2,49,43,11,14,14,50,8. We cubed this product, and it was 1;8,53,32,26,7,0,22,10. Then, we multiplied this cube by 0;0,1,20, and the product was 0;0,1,31,51,23,14,49,20. We added this product to the unmodified [value of the] unknown, and it became even closer [to the value of the]

unknown, [namely,] 1;2,49,43,11,14,44,14,20. Again, we cubed this product, and it was 1;8,53,32,26,8,37,5,34. Then, we multiplied this cube by 0;0,1,20, and the product was 0;0,1,31,51,23,14,51,29. Again, we added this product to the unmodified [value of the] unknown, and the result was what was desired, [namely,] 1;2,49,43,11,14,44,16, for it became fixed at this [value]. This is because if we multiply the cube of this number by 0;0,1,20, and add the product to the unmodified [value of the] unknown, the result will be the same number, without any difference. Thus, this is the Sine of 1°.

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